# Conditions of validity of Čaplygin' s theorem for elliptic difference equations

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## Abstract

The paper considers a necessary and sufficient condition for the validity of a theorem analogous to S. A. Chaplygin' s theorem on differential inequalities for an elliptic difference equation. Effective estimates of the domain of applicability of the mentioned theorem to difference and differential equations are provided. A criterion for the well-posedness of the first boundary value problem for an elliptic difference equation is proven for the case when the known conditions for the applicability of the maximum principle are not satisfied. A criterion for the validity of the maximum principle for an elliptic differential equation, distinct from the known ones, is obtained. Bibliography: 16 items.

## Full Text

## Preamble

The development of numerical methods for differential equations has been a central focus of mathematical research, building on the foundational work of A. L. Souma and others since 1967 [?, ?]. Subsequent studies [?, ?, ?, ?, ?, ?] have expanded these techniques to address complex boundary value problems. We consider a domain $X$ and an $n$-dimensional grid $Q_h$ with step size $h > 0$. Let $x^{(i)}$ denote the coordinates in $n$-dimensional space, where the grid points are defined by $x^{(i)} = k^{(i)}h$ for integers $k^{(i)}$. The boundary of the grid domain is denoted by $\Gamma Q_h$, and the interior points by $VQ_h$.

We define the discrete difference operators as follows: the first-order difference is given by $a_h(k) = \frac{1}{2h}[u_h(k+e_i) - u_h(k-e_i)]$, and the second-order difference (Laplacian) is $\Delta_h u_h(k) = \frac{1}{h^2}\sum_{i=1}^{n}[u_h(k+e_i) - 2u_h(k) + u_h(k-e_i)]$. The general linear difference operator $L_h$ is defined as:

$$L_h[u_h] = \sum_{i=1}^{n} [a_{hi}(k)\Delta_{hi}u_h + b_{hi}(k)\nabla_{hi}u_h] + c_h(k)u_h(k)$$

where the coefficients satisfy the conditions $a_{hi}(k) > m_1 > 0$, $|b_{hi}(k)| < m_2$, and $c_h(k) \leq 0$. These conditions ensure the stability and solvability of the corresponding discrete system [?, ?].

## 1. Discrete Green' s Functions and Stability

For the discrete boundary value problem $L_h[u_h] = f_h(k)$ with boundary conditions $u_h|_{\Gamma Q_h} = \phi_h(k)$, the solution can be expressed using the discrete Green' s function $G_h(k,s)$. The function $G_h(k,s)$ satisfies $L_h[G_h] = \delta(k,s)$, where $\delta$ is the Kronecker delta. Following the methodology in [?, ?, ?, ?], the solution is represented as:

$$u_h(k) = \sum_{s \in VQ_h} G_h(k,s) F_h(s) h^n + \sum_{s \in \Gamma Q_h} \Gamma_h(k,s) \phi_h(s)$$

If $c_h(k) \leq 0$, the maximum principle holds, implying that if $L_h[u_h] \geq 0$ in $VQ_h$ and $u_h \leq 0$ on $\Gamma Q_h$, then $u_h(k) \leq 0$ throughout the domain.

**Lemma 1.** If $c_h(k) \leq 0$ for all $x \in VQ_h$, then the Green' s function $G_h(k,s)$ is non-positive. Furthermore, for two domains $Q_h^{(1)} \subset Q_h^{(2)}$, the corresponding Green' s functions satisfy $G_h^{(1)}(k,s) \leq G_h^{(2)}(k,s)$ for $s \in VQ_h^{(1)}$.

## 2. Spectral Properties and Convergence

When the condition $c_h(k) \leq 0$ is not strictly satisfied, we decompose the coefficient as $c_h(k) = c_h^+(k) - c_h^-(k)$, where both components are non-negative. This allows us to rewrite the operator and analyze the system using an iterative framework $u = A(Q_h)u + f$. The operator $A(Q_h)$ is defined by the kernel $G_h(k,s)c_h^*(s)h^n$. According to the theory of non-negative matrices [?], the spectral radius $r(Q_h)$ determines the convergence of the discrete system.

**Lemma 2.** The spectral radius $r(Q_h)$ is a monotonic function of the domain; specifically, if $Q_h^{(1)} \subset Q_h^{(2)}$, then $r(Q_h^{(1)}) \leq r(Q_h^{(2)})$. This property is critical for establishing the existence of solutions in increasing domains.

## 3. Existence and Uniqueness Theorems

The existence of a solution to the discrete problem (3) is guaranteed if $r(Q_h) < 1$. Under this condition, the operator $[I - A(Q_h)]^{-1}$ exists and is non-negative. This ensures that for any source term $f_h(k)$ and boundary data $\phi_h(k)$, there exists a unique solution $u_h(k)$ that satisfies the discrete equation. If $f_h(k) \leq 0$ and $\phi_h(k) \geq 0$, then $u_h(k) \geq 0$ for all $k \in VQ_h$.

## 4. Convergence to the Continuous Solution

As the grid parameter $h \to 0$, the discrete solution $u_h(k)$ converges to the solution $u(x)$ of the continuous elliptic problem:

$$L[u] = \sum_{i=1}^{n} a_i(x) \frac{\partial^2 u}{\partial x_i^2} + \sum_{i=1}^{n} b_i(x) \frac{\partial u}{\partial x_i} + c(x)u = F(x)$$

with boundary conditions $u|_{\Gamma D} = \phi(x)$. Assuming the coefficients $a_i, b_i, c$ and the source term $F$ are sufficiently smooth and satisfy the ellipticity conditions in domain $D$, the error satisfies $\lim_{h\to 0} \max |u_h(k) - u(x_k)| = 0$.

**Theorem 7.** Let $z(x)$ be a supersolution such that $L[z] \leq F(x)$ and $z|_{\Gamma D} \geq \phi(x)$. Then $z(x) \geq u(x)$ in $D$. This comparison principle extends to the discrete case, providing a robust mechanism for error estimation and stability analysis [**?**, **?**, **?**].

## 5. Conclusion

The results presented demonstrate that the discrete Green' s function method is effective for solving linear elliptic difference equations. By satisfying the spectral radius condition $r(Q_h) < 1$, we ensure the stability of the numerical scheme and its convergence to the continuous solution as the mesh is refined. These findings are consistent with the classical results of Kantorovich, Krylov, and Samarskii regarding the approximation of boundary value problems.

*Note: Figure translations are in progress. See original paper for figures.*

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