Solutions of long wave type for quasilinear elliptic equations in an unbounded strip

Authors: A. M. Ter-Krikorov, V. A. Trenogin

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Abstract

Full Text

Preamble

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LONG-WAVE SOLUTIONS FOR QUASILINEAR ELLIPTIC EQUATIONS IN AN UNBOUNDED STRIP

In hydrodynamics, long waves on the surface of a heavy fluid have been studied for a long time. The solitary wave represents the limiting case of such waves as the wavelength tends to infinity. A mathematically rigorous proof of the existence of a solitary wave was first provided by M. A. Lavrentyev using variational methods he developed within the theory of conformal mappings [?]. K. Friedrichs drew attention to the asymptotic nature of many phenomena in mathematical physics, particularly long and solitary waves [?]. In a joint work by Friedrichs and Hyers [?], these ideas were implemented to prove the existence of a solitary wave; subsequently, one of the authors [?] provided a proof for the existence of solitary and long waves in a heterogeneous swirling fluid.

Solutions of this type may exist not only for the equations of hydrodynamics. It was shown in [?] that solitary-wave solutions exist for a certain class of quasilinear elliptic equations. The corresponding problem for long waves is significantly more difficult. In the present work, this problem is investigated for a somewhat broader class of elliptic equations than in [?].

1. Problem Statement Consider the differential equation in the strip $-\infty < x < +\infty, 0 < y < \pi$:

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + a(y)u = \lambda p(y)u + \Phi(y, u, \nabla u), \quad (1.1)$$

with boundary conditions

$$u(x,0) = u(x,\pi) = 0. \quad (1.2)$$

We seek 2ω-periodic solutions in x for this equation. It is assumed that

$$\Phi(y,u,w) = \sum_{i+j\geq 2} \Phi_{ij}(y) u^i w^j. \quad (1.3)$$

The functions $a(y)$ and $p(y)$ are continuous on $[0,\pi]$, with $p(y) > 0$ on $[0,\pi]$, and the series in formula (1.4) converges for all $y \in [0,\pi]$ in a sufficiently small neighborhood of the origin.

The problem under consideration has a trivial solution for any values of ω and λ. It will be established that there exists a countable set of values for the parameter λ at which small non-trivial solutions can branch off from the trivial solution, degenerating into a solitary wave as $\omega \to \infty$. From hydrodynamics, it is known that long waves are very shallow. Mathematically, this means that differentiation increases the order of smallness of the solution. Based on these heuristic considerations, let us consider the eigenvalues of the following Sturm-Liouville boundary value problem:

$$v'' + a(y)v = \lambda p(y) v, \quad v(0) = v(\pi) = 0. \quad (1.5)$$

It is well known that such a problem has a countable set of eigenvalues, all of which are real and simple. Let λ_0 be one of the eigenvalues of problem (1.5). In equation (1.1), we set:

$$\lambda = \lambda_0 + \mu, \quad x = \xi/\sqrt{|\mu|}, \quad \omega = \Gamma/\sqrt{|\mu|}, \quad \kappa = \operatorname{sgn}(\lambda - \lambda_0). \quad (1.6)$$

Problem (1.1), (1.2) is thus reduced to finding 2Γ-periodic solutions of the equation:

$$\frac{\partial^2 u}{\partial y^2} + [a(y) - \lambda_0 p(y)]u = \lambda_0 \Phi\left(y, u, \frac{\partial u}{\partial \xi}\sqrt{|\mu|}\right) + \kappa\mu p(y) u + \mu\Phi\left(y, u, \frac{\partial u}{\partial \xi}\sqrt{|\mu|}\right) \quad (1.7)$$

under the boundary conditions

$$u(\xi,0) = u(\xi,\pi) = 0. \quad (1.8)$$

Since the boundary conditions and the equation are invariant under a shift in ξ, we can, without loss of generality, seek even periodic solutions of the boundary value problem (1.7), (1.8).

2. Concept of the Generalized Jordan Chain Consider the linear differential operator B, defined on the set of twice continuously differentiable functions $v(y)$ on the interval $[0,\pi]$ that vanish at the endpoints, given by the expression:

$$Bv = \frac{d^2 v}{dy^2} + [a(y) - \lambda_0 p(y)]v. \quad (2.1)$$

Let $\phi_1(y)$ be the eigenfunction of the boundary value problem (1.5) corresponding to the eigenvalue λ_0. Obviously, $B\phi_1 = 0$. It is well known that the inhomogeneous problem $Bv = h(y)$, where $h(y)$ is a continuous function, is solvable if and only if the function $h(y)$ is orthogonal to $\phi_1(y)$ on the interval $[0, \pi]$. The solution to the inhomogeneous problem is defined up to a term $c\phi_1(y)$, where c is an arbitrary constant. If we require the solution to be orthogonal to $\phi_1(y)$, such a solution will be unique.

The Hilbert space $L_2[0, \pi]$ of square-integrable functions $v(y)$ can be represented as a direct sum of two subspaces: $L_2 = E_1 \oplus E_2$, where E_1 is the one-dimensional subspace spanned by the eigenfunction $\phi_1(y)$, and E_2 is its orthogonal complement. Let P and Q be the projection operators. Assuming $(\phi_1, \phi_1) = 1$, we have:

$$Pv = (v, \phi_1)\phi_1(y), \quad Qv = v - Pv. \quad (2.2)$$

Consider the nonlinear operator:

$$\Phi(v) = \lambda_0 \Phi(y, v, 0) \quad (2.3)$$

defined on the set of continuously differentiable functions $v(y)$ on $[0, \pi]$. Let τ and ψ be some formal series in powers of the parameter ϵ, and let $\Phi^{(k)}$ be the coefficient in the formal expansion of the function Φ in powers of ϵ. Using the notation from (2.3) and (2.4), we can write:

$$\Phi\left(\sum \epsilon^k \phi_k\right) = \sum_{k=2}^{\infty} \epsilon^k \Phi^{(k-1)}. \quad (2.5)$$

We shall say that the operator B has a generalized Jordan chain of length p relative to the operator Φ if there exist p linearly independent functions $\phi_1(y), \dots, \phi_p(y)$ that are solutions to the system of equations:

$$B\phi_1 = 0, \quad B\phi_k = Q\Phi^{(k-1)} \quad (k = 2, \dots, p),$$
$$P\Phi^{(1)} = \dots = P\Phi^{(p-1)} = 0, \quad (P\Phi^{(p)}, \phi_1) \neq 0. \quad (2.6)$$

If we require the conditions $(\phi_1, \phi_1) = 1$ and $(\phi_1, \phi_k) = 0$ for $k = 2, \dots, p$, then the Jordan chain is uniquely determined. The concept of a generalized Jordan chain for a linear operator relative to another linear operator was introduced in [**?**]. The generalization to the nonlinear case was presented in [**?**].

Theorem. The solution to the functional equation

$$B\phi = Q\Phi(\phi) \quad (2.7)$$

can be found in the form of a series in powers of a small parameter η:

$$\phi = \eta\phi_1(y) + \eta^2\phi_2(y) + \dots, \quad (\phi_k, \phi_1) = 0 \quad (k = 2, \dots), \quad (2.8)$$

where the functions ϕ_k are twice continuously differentiable on $[0, \pi]$ and vanish at the endpoints. Here, $\phi_1(y), \dots, \phi_p(y)$ are the functions of the generalized

Jordan chain of operator B relative to operator Φ, Q is the projection operator, and the series (2.8) converges uniformly in y on $[0, \pi]$ for values of the parameter $|\eta| < \eta_0$.

Proof. Substituting the series (2.8) into the functional equation (2.7) and equating coefficients of like powers leads to a sequence of linear problems for determining the functions ϕ_k:

$$B\phi_k = Q\Phi^{(k-1)} \quad (k = 2, \dots). \quad (2.9)$$

These problems are solvable because the right-hand sides are orthogonal to $\phi_1(y)$. The convergence of the series (2.8) is proved by constructing majorant series.

3. Reduction of the Boundary Value Problem to a System of Functional Equations Consider the set of continuously differentiable even 2Γ-periodic functions $\tau(\xi)$. We transform this set into a Banach space E_Γ by defining the norm:

$$\|\tau\| = \sup_{\xi} |\tau(\xi)| + \sup_{\xi} |\tau'(\xi)|. \quad (3.1)$$

Consider also the set of continuously differentiable even 2Γ-periodic functions $\psi(\xi, y)$ in the strip $0 < y < \pi$, which are orthogonal to $\phi_1(y)$. We transform this set into a Banach space $E_{\alpha,\Gamma}$ with the norm:

$$\|\psi\| = \sup_{\xi,y} |\psi| + \sup_{\xi,y} |\nabla\psi|. \quad (3.2)$$

Let $B_{\alpha,\Gamma}$ be the space of pairs $\omega = (\tau, \psi)$, where $\tau \in E_\Gamma$ and $\psi \in E_{\alpha,\Gamma}$, with the norm $\|\omega\| = \|\tau\| + \|\psi\|$. Recalling the notation (2.3), we rewrite equation (1.7) as:

$$\frac{\partial^2 u}{\partial \xi^2}\mu + Bu = \mu\kappa p(y)u + [\Phi(u) - \Phi(u)]. \quad (3.4)$$

Let $u(\xi, y) = \phi(y, \tau(\xi)) + \psi(\xi, y)$, where $\phi(y, \tau)$ is the sum of the series (2.8). By the theorem in Section 2, we have:

$$B\phi = Q\Phi(\phi), \quad P\Phi(\phi) = \phi_1(y)\Psi(\tau). \quad (3.6)$$

Substituting this into (3.4) and projecting onto the subspaces E_1 and E_2, we obtain a system of equations for τ and ψ:

$$\mu\frac{d^2\tau}{d\xi^2} - \mu\kappa a\tau - \alpha_{p+1}\tau^{p+1} = Q_1(\omega), \quad (3.8)$$

$$\mu\frac{\partial^2\psi}{\partial\xi^2} + B\psi = Q_2(\omega), \quad (3.9)$$

where Q_1 and Q_2 are analytic operators in a neighborhood of $\omega = 0$. For these operators, estimates of the form $\|Q\| < C(|\mu| + \|\tau\|^r)$ hold, ensuring the validity of the iterative process.

4. Construction of the Approximate Solution We seek a formal solution to the system (3.8), (3.9) in the form of series in fractional powers of the small parameter μ:

$$\tau = \mu^{1/p} \sum \mu^{k/p} \tau_k, \quad \psi = \mu \sum \mu^{k/p} \psi_k. \quad (4.1)$$

From (3.8), the leading term τ_0 satisfies a nonlinear ordinary differential equation:

$$\frac{d^2\tau_0}{d\xi^2} = \kappa a \tau_0 + \alpha_{p+1} \tau_0^{p+1}. \quad (4.2)$$

As shown in Appendix I, for p odd, this equation has at least one 2Γ-periodic solution that degenerates into a solitary wave as $\Gamma \to \infty$. For p even, the existence of solutions depends on the sign of α_{p+1} and κ. Each 2Γ-periodic solution of (4.2) corresponds to a unique formal 2Γ-periodic solution of the system, and thus of the original boundary value problem.

5. Existence Theorem and Error Estimates **Existence Theorem.** Let the Jordan chain of operator B relative to Φ have length p. If a formal solution to the system (3.8), (3.9) can be constructed as series (4.1), then for sufficiently small μ, there exists an exact solution in $B_{\alpha,\Gamma}$ for which (4.1) serves as an asymptotic expansion as $\mu \to 0$. The difference between the exact solution and a finite truncation of the asymptotic series tends to zero faster than the last included term.

The proof utilizes the contraction mapping principle in Banach spaces. By defining the operators L and M associated with the linearized versions of (3.8) and (3.9), and proving their invertibility (as shown in Appendices II and III), the problem is reduced to a fixed-point equation $\omega^* = \Omega(\omega^*)$. The uniformity of the estimates with respect to the parameter Γ ensures that the solutions persist as $\Gamma \to \infty$, where they degenerate into solitary waves.

Remark. These results can be extended to a more general class of elliptic equations in cylindrical domains $\Omega \times (-\infty, +\infty)$ of the form:

$$\Delta u + a(y, u, \nabla u) = \lambda F(y, u, \nabla u), \quad (5.4)$$

provided the ellipticity condition is satisfied and the corresponding linear problem possesses at least one eigenvalue.

Appendix I: Investigation of Periodic Solutions Consider the equation:

$$\frac{d^2\zeta}{dt^2} = a\zeta + b\zeta^{p+1}. \quad (1)$$

For p odd, there exists at least one even $2T$-periodic solution for $T > \pi/\sqrt{|a|}$, which degenerates into a solitary wave as $T \to \infty$. The period T is expressed via the integral:

$$T = \int_0^{u_0} \frac{du}{\sqrt{au^2 + \frac{2b}{p+2} u^{p+2} + C}}. \quad (5)$$

As the integration constant $C \to 0$, the period $T \to \infty$. For $p = 1$, the solution is expressed in terms of elliptic functions, and the monotonicity of $T(C)$ can be rigorously proven.

Appendix II: Linear Equation Analysis The linear inhomogeneous equation $y'' - [a - (q+1)b\tau_0^q]y = f$ is studied. It is shown that for sufficiently large T, the operator has a bounded inverse in the space of periodic functions. The proof relies on Liouville' s formula and asymptotic estimates of the fundamental solutions.

Appendix III: Higher-Order Terms The existence of a bounded inverse for the operator acting on the ψ component is established using the Green' s function for the Sturm-Liouville operator. The estimates are obtained via the "parametrix" method, ensuring that the iterative procedure for the full nonlinear system converges in the appropriate Banach space.

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Note: Figure translations are in progress. See original paper for figures.