

# ON THE QUESTION OF UNIFORM APPROXIMATION OF FUNCTIONS BY POSITIVE LINEAR OPERATORS

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**Abstract**

**Full Text**

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*MATHEMATICS*

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## ON THE QUESTION OF UNIFORM APPROXIMATION OF FUNCTIONS BY POSITIVE LINEAR OPERATORS

*(Presented by Academician V. I. Smirnov on 1 IV 1966)*

Let  $a$  and  $b$ ,  $a < b$ , be fixed numbers; let  $\omega(h)$ ,  $0 \leq h \leq b - a$ ,  $\omega(0) = 0$ , be a nondecreasing function, positive for  $0 < h \leq b - a$ ; let  $H^\omega$  be the class of functions  $f(x)$ ,  $a \leq x \leq b$ , such that

$$|f(x_1) - f(x_2)| \leq \omega(|x_1 - x_2|)$$

for  $x_1$  and  $x_2$  from  $[a, b]$ ; let  $\sigma(x)$ ,  $a \leq x \leq b$ , be a nondecreasing function having an infinite number of points of increase and such that  $\sigma(b) - \sigma(a) = 1$ ; let  $\{\varphi_k(x)\}_0^\infty$  be a system of continuous functions orthonormal on  $[a, b]$  with integral weight  $\sigma(x)$  (see (1), pp. 134, 144), with  $\varphi_0(x) \equiv 1$ , and let  $\lambda = \{\lambda_k\}_0^\infty$  be a collection of real numbers  $\lambda_k$  such that the series

$$K(x, t, \lambda) = \sum_{k=0}^{\infty} \lambda_k \varphi_k(x) \varphi_k(t), \quad a \leq x, t \leq b,$$

converges uniformly and its sum  $K(x, t, \lambda) \geq 0$ ,  $a \leq x, t \leq b$ .

Let  $f(x) \in H^\omega$ , and let  $a_k$  be the Fourier coefficients of  $f(x)$ :

$$a_k = \int_a^b f(t) \varphi_k(t) d\sigma(t), \quad k = 0, 1, \dots$$

Introduce the positive linear operator

$$U(x, f, \lambda) = \sum_{k=0}^{\infty} \lambda_k a_k \varphi_k(x) = \int_a^b f(t) K(x, t, \lambda) d\sigma(t).$$

In what follows we put  $\lambda_0 = 1$ . Then

$$f(x) - U(x, f; \lambda) = \int_a^b [f(x) - f(t)] K(x, t, \lambda) d\sigma(t).$$

Hence, for

$$E(x, \lambda) = \sup_{f \in H^\omega} |f(x) - U(x, f, \lambda)|$$

we obtain the estimate

$$E(x, \lambda) \leq \int_a^b \omega(|x-t|) K(x, t, \lambda) d\sigma(t) = \sum_{k=0}^{\infty} \lambda_k a_k(x) \varphi_k(x),$$

$$a_k(x) = \int_a^b \omega(|x-t|) \varphi_k(t) d\sigma(t). \quad (1)$$

If, for some  $x = x_0$ ,  $a \leq x_0 \leq b$ , the function  $\omega(|x_0 - x|) \in H^\omega$ , then in estimate (1) at  $x = x_0$  equality holds. In particular, if  $\omega(x)$  is a modulus of continuity, then equality in (1) holds for every  $x$ ,  $a \leq x \leq b$ .

**Example.** Let  $a = -1$ ,  $b = 1$ ;  $\omega(h) = h$ ;  $\sigma(x) = 1/\pi \arcsin x$ ;  $\varphi_k(x) = \sqrt{2} \cos k \arccos x = \sqrt{2} T_k(x)$ ,  $k = 1, 2, \dots$ . In this case

$$K(\cos \theta, \cos \varphi, \lambda) = 1 + 2 \sum_{k=1}^{\infty} \lambda_k \cos k\theta \cos k\varphi =$$

$$= \frac{1}{2} K(\theta - \varphi, \lambda) + \frac{1}{2} K(\theta + \varphi, \lambda),$$

$$K(\theta, \lambda) = 1 + 2 \sum_{k=1}^{\infty} \lambda_k \cos k\theta,$$

and, consequently, in order that the function  $K(x, t, \lambda)$  be nonnegative for  $-1 \leq x, t \leq 1$ , it is necessary and sufficient that the function  $K(\theta, \lambda)$  be nonnegative for  $0 \leq \theta < \pi$ .

Let us find  $E(x, \lambda)$  for this case. Putting  $x = \cos \theta$ ,  $t = \cos \varphi$ ,  $0 \leq \theta, \varphi \leq \pi$ , we have

$$a_0(x) = \frac{1}{\pi} \int_a^b |x-t| \frac{dt}{\sqrt{1-t^2}} = \frac{1}{\pi} \int_0^\pi |\cos \theta - \cos \varphi| d\varphi =$$

$$= \left(1 - \frac{2\theta}{\pi}\right) \cos \theta + \frac{2}{\pi} \sin \theta,$$

$$a_1(x) = -\frac{1}{\sqrt{2}} \left(1 - \frac{2\theta}{\pi}\right) - \frac{1}{\sqrt{2}\pi} \sin 2\theta,$$

$$a_k(x) = \frac{\sqrt{2}}{\pi} \left[ \frac{1}{k(k-1)} \sin(k-1)\theta - \frac{1}{k(k+1)} \sin(k+1)\theta \right], \quad k = 2, 3, \dots,$$

and formula (1) gives

$$E(x, \lambda) = (1 - \lambda_1) \left( 1 - \frac{2}{\pi} \theta \right) \cos \theta + \frac{1}{\pi} \left[ 2 \left( 1 - \frac{\lambda_1}{4} - \sum_{k=2}^{\infty} \frac{\lambda_k}{k^2 - 1} \right) - \sum_{k=1}^{\infty} \frac{\lambda_k - \lambda_{k+1}}{k(k+1)} \sin(2k+1)\theta \right].$$

Returning to the variable  $x$ , we finally obtain

$$E(x, \lambda) = (1 - \lambda_1)x \left( 1 - \frac{2}{\pi} \arccos x \right) + \frac{1}{\pi} \sqrt{1 - x^2} \left[ 2 \left( 1 - \frac{\lambda_1}{4} - \sum_{k=2}^{\infty} \frac{\lambda_k}{k^2 - 1} \right) - \sum_{k=1}^{\infty} \frac{\lambda_k - \lambda_{k+1}}{k(k+1)(2k+1)} T'_{2k+1}(x) \right].$$

If here we put  $\lambda_k = (n - k)/n$ ,  $k = 1, 2, \dots, n - 1$ ,  $\lambda_k = 0$ ,  $k = n, n + 1$ , then we arrive at the result of A. F. Timan<sup>2</sup> (obtained by him by another method) for the arithmetic means of the Fourier-Chebyshev series.

In conclusion I express my deep gratitude to N. A. Lebedev for useful advice.

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## REFERENCES

- <sup>1</sup> V. L. Goncharov, *Theory of Interpolation and Approximation of Functions*, 1954.  
<sup>2</sup> A. F. Timan, DAN, 77, No. 6, 969 (1951).

*Note: Figure translations are in progress. See original paper for figures.*

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