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Abstract

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MATHEMATICS

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ON THE UNIFORM CONVERGENCE OF FOURIER SERIES IN JACOBI POLYNOMIALS

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Let the function $f(x)$ be continuous and have bounded variation on the interval $[-1,1]$, and let the series

$$\sum_{k=0}^{\infty} c_k \hat{P}_k^{(\alpha,\beta)}(x) \tag{1}$$

be its Fourier series in Jacobi polynomials, orthonormal on the interval $[-1,1]$ with weight $p(x)=(1-x)^{\alpha}(1+x)^{\beta}$. From the equiconvergence theorem of G. Szegő ([5], p. 254) it follows that for $\alpha>-1,\ \beta>-1$ the series (1) converges uniformly on the interval $[-1+\mu,\,1-\mu]$, where $0<\mu<1$. In the case of Legendre polynomials ($\alpha=\beta=0$), E. V. Hobson ([4], p. 325) also established that the series (1) converges at the points $x=\pm 1$; if, moreover, $-1<\alpha<-1/2$ and $-1<\beta<-1/2$, then the convergence of the series (1) at these points follows from the results of G. Rau ([1]).

In the present paper, the uniform convergence of the series (1) is established on the whole interval $[-1,1]$ under the conditions $-1<\alpha<1/2,\ -1<\beta<1/2$, and $|\alpha-\beta|<1$ (as is known, the case $\alpha=\beta=-1/2$ has been well studied).

Introduce the notation

$$K_n^{(\alpha,\beta)}(t,x)=\sum_{k=0}^{n}\hat{P}_k^{(\alpha,\beta)}(t)\hat{P}_k^{(\alpha,\beta)}(x). \tag{2}$$

Lemma 1. *For the kernel (2) the following formula holds*

$$K_n^{(\alpha,\beta)}(t,x)=(1-x^2)K_{n-1}^{(\alpha+1,\beta+1)}(t,x)+ \tag{3}$$

$$+\theta_n^{(\alpha,\beta)}\left[(n+\alpha+\beta+2)P_n^{(\alpha+1,\beta+1)}(t)P_n^{(\alpha,\beta)}(x)+(n+1)P_{n-1}^{(\alpha+1,\beta+1)}(t)P_{n+1}^{(\alpha,\beta)}(x)\right],$$

where

$$P_n^{(\alpha,\beta)}(x) = \sqrt{\frac{2^{\alpha+\beta+1}\Gamma(n+\alpha+1)\Gamma(n+\beta+1)}{(2n+\alpha+\beta+1)\Gamma(n+\alpha+\beta+1)\Gamma(n+1)}}\,\hat{P}_n^{(\alpha,\beta)}(x), \quad (4)$$

$$\theta_n^{(\alpha,\beta)} = O(1), \qquad K_{-1}^{(\alpha+1,\beta+1)}(t,x) \equiv 0, \qquad P_{-n}^{(\alpha+1,\beta+1)}(t) \equiv 0.$$

Formula (3) is a consequence of known relations for Jacobi polynomials ([4]), contained in ([5]), Ch. IV.

Put

$$\Phi_n^{(\alpha,\beta)}(a,b;x) = \int_a^b p(t)K_n^{(\alpha,\beta)}(t,x)\,dt, \qquad -1 \le a < b \le 1, \quad x \in [-1,1], \quad (5)$$

and by $\Phi_n^{(\alpha,\beta)}(E;x)$ we shall denote the same integral, taken over a set $E \subset [-1,1]$. In view of the validity of the formula ([5], p. 71)

$$P_n^{(\alpha,\beta)}(x) = (-1)^n P_n^{(\beta,\alpha)}(-x) \quad (6)$$

it is sufficient to consider the case when $x \in [0,1]$.

Lemma 2. Let $-\frac{1}{2} \le \alpha \le \frac{1}{2}$, $\beta > -1$, and $x \in [0,1]$. Then, provided the condition $\alpha - \beta \le 1$ is satisfied, there exists a constant C_1, depending only on α and β, such that for all n the inequality

$$\left|\Phi_n^{(\alpha,\beta)}(a,b;x)\right| \le C_1, \qquad -1 \le a < b \le 1. \quad (7)$$

holds.

Proof. Let $E_1 = \{t;\ |t-x| < \delta\}$, where $0 < \delta < 1$. We first consider the case when $[a,b] \cap E_1 = \emptyset$. Using formula ((5), p. 83)

$$K_n^{(\alpha,\beta)}(t,x) = l_n^{(\alpha,\beta)}\frac{P_{n+1}^{(\alpha,\beta)}(t)P_n^{(\alpha,\beta)}(x) - P_n^{(\alpha,\beta)}(t)P_{n+1}^{(\alpha,\beta)}(x)}{t-x}, \quad (8)$$

where $l_n^{(\alpha,\beta)} = O(n)$, and the second mean-value theorem, from (5) we obtain

$$\left|\Phi_n^{(\alpha,\beta)}(a,b;x)\right| \le C_2\delta^{-1}n\int_{a_1}^{b_1} p(t)\left[P_{n+1}^{(\alpha,\beta)}(t)P_n^{(\alpha,\beta)}(x) - P_n^{(\alpha,\beta)}(t)P_{n+1}^{(\alpha,\beta)}(x)\right]dt. \quad (9)$$

where $[a_1, b_1] \subset [a, b]$. For the polynomials (4) the following relation holds ((5), p. 175)

$$\max_{x\in[0,1]} \left|P_n^{(\alpha,\beta)}(x)\right| = \left|P_n^{(\alpha,\beta)}(1)\right| = \frac{\Gamma(n+\alpha+1)}{\Gamma(\alpha+1)\Gamma(n+1)}, \qquad \alpha \geq -\frac{1}{2}, \tag{10}$$

and the inequality

$$(1-x)^{\alpha/2+1/4}(1+x)^{\beta/2+1/4}\left|P_n^{(\alpha,\beta)}(x)\right| \leq \frac{C_3}{\sqrt{n}},$$

$$\min(\alpha,\beta) = \rho \geq -\frac{1}{2}, \qquad x \in [-1,1], \tag{11}$$

which is a consequence of formula 7.32.5 ((5), p. 176). If $\beta \geq -\frac{1}{2}$, then by means of (10), (11), and formula ((5), p. 107)

$$(1-x)^{\alpha}(1+x)^{\beta}P_n^{(\alpha,\beta)}(x) = -\frac{1}{2n}\frac{d}{dx}\left\{(1-x)^{\alpha+1}(1+x)^{\beta+1}P_{n-1}^{(\alpha+1,\beta+1)}(x)\right\} \tag{12}$$

from (9) we find

$$\left|\Phi_n^{(\alpha,\beta)}(a,b;x)\right| \leq C_4\delta^{-1}n^{\alpha-1/2}. \tag{13}$$

If $-1 < \beta < -\frac{1}{2}$, then instead of (11) we use the inequality of G. Szegő ([2])

$$(1+x)^{\lambda}\left|P_n^{(\alpha,\beta)}(x)\right| \leq C_5 n^{\beta-2\lambda}, \tag{14}$$

where $\beta > -\frac{1}{2}$, $0 \leq \lambda < \beta/2 + \frac{1}{4}$, $x \in [-1, 1-\mu]$, and obtain

$$\left|\Phi_n^{(\alpha,\beta)}(a,b;x)\right| \leq C_6\delta^{-1}n^{\alpha-\beta-1}. \tag{15}$$

Now let $[a,b] \cap E_1 = E_2$ and $E_2 \neq \emptyset$, while $E_3 = [a,b] \setminus E_2$. For $\Phi_n^{(\alpha,\beta)}(E_3;x)$ estimates analogous to (13) or (15) hold. To obtain an estimate for $\Phi_n^{(\alpha,\beta)}(E_2;x)$, we make the substitution $t = \cos\theta$. To the set E_2 there corresponds a set $D \subset [0,\pi]$, which we split into parts: $D_1 = \{\theta;\ |\theta-\theta_0| \leq 1/n\}$ and $D_2 = D \setminus D_1$; here $\theta_0 = \arccos x$.

a) Let $D_1 \neq \emptyset$. From (10) and (11) we obtain

$$\left|P_n^{(\alpha,\beta)}(\cos\theta)\right| \le C_7\theta^{-\alpha-1/2}[1-\varphi_n(\theta)]n^{-1/2} + C_8 n^\alpha \varphi_n(\theta), \tag{16}$$

$$\alpha \ge -\frac{1}{2}, \qquad 0 \le \theta \le \pi/2, \qquad \varphi_n(\theta) = \begin{cases} 1, & 0 \le \theta \le n^{-1}, \\ 0, & n^{-1} < \theta \le \pi/2. \end{cases}$$

Taking into account (4), (5), and (16), we arrive at the inequality

$$\left|\Phi_n^{(\alpha,\beta)}(D_1;\cos\theta_0)\right| \le C_9\left[\int_{D_1}\theta^{2\alpha+1}\,d\theta\right]^{1/2}\left[\sum_{k=0}^{n}\left[\widehat{P}_k^{(\alpha,\beta)}(\cos\theta_0)\right]^2\right]^{1/2} \le$$

$$\le C_{10}\left[\int_{D_1}\sum_{k=0}^{n}\left\{[1-\varphi_k(\theta_0)]\left(\frac{\theta}{\theta_0}\right)^{\alpha+1/2} + (k\theta)^{\alpha+1/2}\varphi_k(\theta_0)\right\}^2 d\theta\right]^{1/2} \le C_{11}. \tag{17}$$

b) Let, next, $D_2^{(1)} = D_2 \cap (\theta_0 + n^{-1}, \arccos(x-\delta))$. Then, using formula (3) and introducing for the integrals obtained the notation I_1, I_2, I_3, we have

$$\begin{aligned} \left|\Phi_n^{(\alpha,\beta)}(D_2^{(1)};\cos\theta_0)\right| \le C_{12}\Bigg\{&\left|\int_{D_2^{(1)}}\left(\sin\frac{\theta}{2}\right)^{2\alpha+1}\left(\cos\frac{\theta}{2}\right)^{2\beta+1}\sin^2\theta_0\, K_{n-1}^{(\alpha+1,\beta+1)}(\cos\theta,\cos\theta_0)\,d\theta\right| \\ &+\left|\int_{D_2^{(1)}}\left(\sin\frac{\theta}{2}\right)^{2\alpha+1}\left(\cos\frac{\theta}{2}\right)^{2\beta+1} nP_n^{(\alpha,\beta)}(\cos\theta_0)P_n^{(\alpha+1,\beta+1)}(\cos\theta)\,d\theta\right| \\ &+\left|\int_{D_2^{(1)}}\left(\sin\frac{\theta}{2}\right)^{2\alpha+1}\left(\cos\frac{\theta}{2}\right)^{2\beta+1} nP_{n+1}^{(\alpha,\beta)}(\cos\theta_0)P_{n-1}^{(\alpha+1,\beta+1)}(\cos\theta)\,d\theta\right|\Bigg\} \\ = C_{12}\{I_1+I_2+I_3\}.& \end{aligned} \tag{18}$$

To estimate I_1 we use inequality (11), the asymptotic formula ((5), p. 205)

$$P_n^{(\alpha,\beta)}(\cos\theta) = n^{-1/2}k(\theta)\{\cos(N\theta+\gamma) + (n\sin\theta)^{-1}O(1)\}, \tag{19}$$

where

$$k(\theta) = \pi^{-1/2}\left(\sin\frac{\theta}{2}\right)^{-\alpha-1/2}\left(\cos\frac{\theta}{2}\right)^{-\beta-1/2}, \quad N = n + \frac{\alpha+\beta+1}{2},$$

$$\gamma = -(\alpha+1/2)\pi/2, \quad \alpha > -1,\ \beta > -1, \quad cn^{-1} \le \theta \le \pi - cn^{-1},$$

$c > 0$ is a fixed number, and the identity ((5), p. 83)

$$\frac{2n+\alpha+\beta+2}{2}(1-x)P_n^{(\alpha+1,\beta)}(x) =$$

$$= (n+\alpha+1)P_n^{(\alpha,\beta)}(x) - (n+1)P_{n+1}^{(\alpha,\beta)}(x). \tag{20}$$

After transformations from (18) we obtain

$$|I_1| \le C_{13}\left\{ \left| \int_{D_2^{(1)}} \left(\frac{\theta_0}{\sin\frac{\theta}{2}}\right)^{1/2-\alpha} \frac{\theta_0[\cos(N_1\theta+\gamma_1)+(n\sin\theta)^{-1}]}{\left(\cos\frac{\theta}{2}\right)^{1/2-\beta}\sin\frac{\theta-\theta_0}{2}\,\sin\frac{\theta+\theta_0}{2}}\,d\theta \right| \right.$$
$$\left. + \int_{D_2^{(1)}} \left(\frac{\theta_0}{\sin\frac{\theta}{2}}\right)^{1/2-\alpha} \frac{\sin\frac{\theta}{2}[\cos(N_2\theta+\gamma_2)+(n\sin\theta)^{-1}]}{\left(\cos\frac{\theta}{2}\right)^{1/2-\beta}\sin\frac{\theta-\theta_0}{2}\,\sin\frac{\theta+\theta_0}{2}}\,d\theta \right\},$$

whence, after applying the second mean-value theorem, the estimate $|I_1| \le C_{14}$ follows. With the help of (10) and (19) it is also not difficult to establish the boundedness of the integrals I_2 and I_3 in (18), which leads to the inequality

$$\left|\Phi_n^{(\alpha,\beta)}(D_2^{(1)};\cos\theta_0)\right| \le C_{15}.$$

c) Let, finally, $D_2^{(2)} = D_2 \setminus D_2^{(1)}$ and $D_2^{(2)} \ne \emptyset$. If $1/n \in D_2^{(2)}$, then, arguing as in case a), we find the inequality $\left|\Phi_n^{(\alpha,\beta)}(\cos 1/n, b; x)\right| \le C_{16}$. The further arguments are carried out with the aid of relations (8), (11), (12), (19), and (20), after which we obtain the estimate

$$\left|\Phi_n^{(\alpha,\beta)}(D_2^{(2)};\cos\theta_0)\right| \le C_{17},$$

and Lemma 2 is proved.

Let us note that inequality (7) in the case when $\alpha = \beta = 1/2$ (Chebyshev polynomials of the second kind) is not difficult to verify directly.

Lemma 3. *If $-1 < \alpha < -1/2$ and $\beta > -1$, then there exists a constant C_{18}, depending on α and β, such that for all $x \in [0,1]$ and all n the inequality holds*

$$\left|\Phi_n^{(\alpha,\beta)}(a,b;x)\right| \le C_{18}, \qquad -1 \le a < b \le 1. \tag{21}$$

The proof is essentially the same as the preceding one. To estimate $\Phi_n^{(\alpha,\beta)}(E_2;x)$, the relation

$$\Phi_n^{(\alpha,\beta)}(E_2;x) = \Phi_{n-1}^{(\alpha+1,\beta+1)}(E_2;x) + \\ + \theta_n^{(\alpha,\beta)} \int_{E_2} p(t)\big[(n+\alpha+\beta+2)P_n^{(\alpha+1,\beta+1)}(x)P_n^{(\alpha,\beta)}(t) + \\ + (n+1)P_{n-1}^{(\alpha+1,\beta+1)}(x)P_{n+1}^{(\alpha,\beta)}(t)\big]\,dt,$$

which is a consequence of (3), is additionally used.

Thus, under the conditions $-1 < \alpha \le 1/2$, $-1 < \beta \le 1/2$, and $|\alpha - \beta| \le 1$, Lemmas 2, 3 and relation (7) imply the inequality

$$\left|\Phi_n^{(\alpha,\beta)}(a,b;x)\right| \le C_{19}, \qquad -1 \le a < b \le 1, \qquad -1 \le x \le 1. \tag{22}$$

Theorem. *If $-1 < \alpha < 1/2$, $-1 < \beta < 1/2$, and $|\alpha - \beta| < 1$, then every continuous function $f(x)$ of bounded variation on the interval $[-1,1]$ is expanded in a Fourier–Jacobi series converging uniformly on this interval.*

Proof. It is enough to consider the case where $f(x)$ is continuous and nondecreasing on the interval $[-1,1]$. As is known (3), the remainder of the series (1) may be represented in the form

$$R_n^{(\alpha,\beta)}(f;x) = \int_{-1}^{1} [f(x) - f(t)]\, p(t)\, K_n^{(\alpha,\beta)}(t,x)\, dt,$$

and the validity of the assertion of the theorem follows from relations (8), (22), (11) or (14).

In conclusion we give two monotone continuous functions with divergent Fourier–Jacobi series under some conditions on α and β. Let $f_1(x) = \sqrt{1+x}$ for $x \in [-1,0]$, and $f_1(x) \equiv 1$ for $x \in (0,1]$. Then for $\alpha = \beta = 1/2$ we obtain

$$R_n^{(1/2,1/2)}(f;1) = -(-1)^{[(n+1)/2]}/\pi.$$

If, however, we put

$$f_2(x) = (1+x)^{(\alpha-\beta-1)/2},$$

where $\alpha - \beta > 1$, then, using a result of G. Szegő ((5), p. 264), we find that the Fourier–Jacobi series of this function also diverges at the point $x = 1$.

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Note: Figure translations are in progress. See original paper for figures.