AN INTEGRAL OF TEMLYAKOV TYPE AND ITS LIMITING VALUES

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AN INTEGRAL OF TEMLYAKOV TYPE AND ITS LIMITING VALUES

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1. In the present work a new definition is given of an integral of Temlyakov type, and its limiting values at points of the neighborhoods $B_{s,j}$ are studied; it is established that, generally speaking, they differ on certain two-dimensional surfaces. We note that, although the work was written jointly, the result on limiting values belongs to the first author, and the concept of an integral of Temlyakov type to the second.

2. Let D be a bicircular domain of class (T) (([4]), p. 349), i.e., a complete, bounded, convex bicircular domain with center at the point $(0, 0) \in D$, whose boundary is twice continuously differentiable and analytically convex from the outside. As is known ([1–3]), a domain $D \in (T)$ can be given as a bicircular domain with center at the point $(0, 0) \in D$ of the space C^2 of complex variables (w, z), bounded by the nonanalytic hypersurface

$$|w| = r_1(\tau),$$

and

$$|z| = r_2(\tau) \qquad (0 \le \tau \le 1), \tag{1}$$

where

$$r_1(0) = 0, \qquad (r_1(\tau)/\tau)' \le 0 \qquad (0 < \tau \le 1), \tag{2}$$

$$r_2(\tau) = \exp\left[-\int_0^\tau t(1-t)\, d\ln r_1(\tau)\right]. \tag{3}$$

Remark 1. Conditions (2) and (3) are equivalent to the requirement that the curve defined by the equations $|w| = r_1(\tau)$ and $|z| = r_2(\tau)$ $(0 \le \tau \le 1)$, and which is the boundary of the image of the domain D in the absolute quadrant-plane, is the envelope of the family of straight lines

$$c(\tau)|w| + d(\tau)|z| = 1,$$

where

$$c(\tau) = \tau r_1^{-1}(\tau), \qquad d(\tau) = (1-\tau) r_2^{-1}(\tau) \qquad (0 \le \tau \le 1),$$

and lies below the envelope at every point.

Consider a function $\Phi(\tau, t, \lambda, \mu)$ (τ and t are real variables, λ and μ complex), which is summable in the rectangle

$$R = \{\tau, t: \ 0 \le \tau \le 1, \ 0 \le t \le 2\pi\}$$

for arbitrary λ and μ ($|\lambda| < +\infty, \ |\mu| < +\infty$).

We shall call an **integral of Temlyakov type** the integral

$$f(w, z) = (4\pi^2 i)^{-1} \int_0^1 d\tau \int_0^{2\pi} dt \int_{|\xi|=1} \Phi(\tau, t, \xi, \eta)(\xi - u)^{-k} \, d\xi, \tag{4}$$

where $k = 1, 2$; $(w, z) \in C^2$; $\eta = \xi e^{-it}$; $u = c(\tau)w + d(\tau)ze^{it}$; $r_1(\tau)$ and $r_2(\tau)$ are functions determined by relations (2) and (3); moreover, if $k = 1$, then the integral (4) will be called an **integral of Temlyakov type of the first kind**; if $k = 2$, an **integral of Temlyakov type of the second kind**. Since the functions $r_i = r_i(\tau)$ $(i = 1, 2)$ are determined by relations (2) and (3), one may consider the nonanalytic hypersurface

$$|w| = r_1(\tau), \qquad |z| = r_2(\tau) \qquad (0 \le \tau \le 1). \tag{5}$$

which is the envelope of the family of hyperplanes $c(\tau)|w| + d(\tau)|z| = 1$ $(0 \le \tau \le 1)$ and is situated under the envelope at each point. According to Remark 1, the nonanalytic hypersurface (5) can be regarded as the boundary of the complete, convex, bicircular domain

$$D = \{w, z: \ |w| < r_1(\tau), \ |z| < r_2(\tau), \ 0 \le \tau \le 1\},$$

whose boundary is twice continuously differentiable and analytically convex from the outside, i.e. $D \in (T)$.

Remark 2. In the integral (4) the density $\Phi(\tau, t, \xi, \xi e^{-it})$ is given on the topological product

$$M = \{\tau, t, \xi: \ 0 \le \tau \le 1, \ 0 \le t \le 2\pi, \ |\xi| = 1\};$$

we note, moreover, that the density $\Phi(\tau, t, \xi, \zeta e^{-it})$ can be given on a manifold of two or three dimensions of the space C^2, lying either in the domain D, or outside it, on the boundary ∂D of the domain D. The case of an integral of Temlyakov type in which the density is given on the boundary ∂D of the domain D was considered earlier in the works ([5–7]).

Let the curve $|z| = \Psi(|w|)$ (or $r_2(\tau) = \Psi(r_1(\tau))$) be the image of the boundary ∂D of the domain D in the absolute quarter-plane. We shall denote by $[\alpha_j, \beta_j]$ $(j = 1, 2, \dots, n)$ the intervals of variation of the parameter τ such that the function $\Psi(r_1(\tau))$ is linear on the intervals $[r_1(\alpha_j), r_1(\beta_j)]$ $(j = 1, 2, \dots, n)$, where $r_1(\tau) = \tau c_j^{-1}$ $(c_j > 0)$ for $\tau \in [\alpha_j, \beta_j]$ $(j = 1, 2, \dots, n)$. As is known ([7]), the **order of the boundary** $|z| = \Psi(|w|)$ $(r_2(\tau) = \Psi[r_1(\tau)])$ of the domain D is called the number of intervals of linearity $[r_1(\alpha_j), r_1(\beta_j)]$ of the function $\Psi(r_1(\tau))$ belonging to the segment $[0, r_1(1)]$.

Definition 1. We shall say that $\Phi(\tau, t, \xi, \eta) \in \lambda$ $(\eta = \zeta e^{-it})$, if $\Phi(\tau, t, \xi, \eta)$ is a summable function on the set

$$R = \{\tau, t: \ 0 \le \tau \le 1, \ 0 \le t \le 2\pi\}$$

for arbitrary ξ and η $(|\xi| = |\eta| = 1)$, and, with respect to the variable ξ, satisfies the condition $\operatorname{Lip}\alpha$ $(0 < \alpha \le 1)$, independent of τ and t (i.e.

$$|\Phi(\tau, t, \xi, \zeta e^{-it}) - \Phi(\tau, t, \xi_0, \zeta_0 e^{-it})| < N|\xi - \xi_0|^{\alpha},$$

where N and α do not depend on τ and t).

Theorem 1. *Let the density of the integral of Temlyakov type (4) be the function* $\Phi(\tau, t, \xi, \zeta e^{-it}) \in \lambda$,

$$a = \lim_{\tau \to 0} \frac{r_2'(\tau)}{r_1'(\tau)}, \qquad b = \lim_{\tau \to 1} \frac{r_2'(\tau)}{r_1'(\tau)}.$$

Then, if $a \ne 0$ and $b \ne -\infty$ (or $a \ne 0$ and $b = -\infty$, or $a = 0$ and $b \ne -\infty$, or $a = 0$ and $b = -\infty$), the integral (4) is a function holomorphic in the domains D, E_1, and E_2 (or in D and E_1, or in D and E_2, or in D) and nonholomorphic in the domain $C^2 \setminus \overline{D \cup E_1 \cup E_2}$ (or $C^2 \setminus \overline{D \cup E_1}$, or $C^2 \setminus \overline{D \cup E_2}$, or $C^2 \setminus \overline{D}$), where

$$E_1 = \{w, z: \ f_1(|w|, |z|, 0) < 0\}, \qquad E_2 = \{w, z: \ f_2(|w|, |z|, 1) > 0\},$$
$$f_1(|w|, |z|, \tau) = |z| + r_2'(\tau)(r_1'(\tau))^{-1}|w| - r_2'(\tau)(r_1'(\tau))^{-1} r_1(\tau) + r_2(\tau),$$
$$f_2(|w|, |z|, \tau) = f_1(|w|, |z|, \tau) - 2\big[r_2(\tau) - r_2'(\tau) r_1(\tau)(r_1'(\tau))^{-1}\big].$$

Definition 2. We shall say that $\Phi(\tau, t, \xi, \eta) \in a$ $(\eta = \zeta e^{-it})$, if $\Phi(\tau, t, \xi, \eta)$ satisfies, with respect to ξ, the condition $\operatorname{Lip}\alpha$ $(0 < \alpha \le 1)$, independent of τ and t, and is bounded in modulus for $(\tau, t, \xi) \in M$.

Theorem 2. *Let the boundary of the domain D have order n, and let the density of the integral (4) be a function $\Phi(\tau, t, \xi, \eta) \in a$. Then the integral (4) is continuous in the whole space C^2, with the possible exception of the circles*

$$B_{s,j} = \{w, z: \ |w| = c_j^{-1}, \ |z| = 0, \ c_j > 0, \ s = -1 \text{ or } |w| = 0, \ |z| = d_j^{-1}, \ d_j > 0, \ s = +1, \ j = 1, 2, \dots, n\}.$$

Fig. 1

Figure 1: Fig. 1

Remark 3. Since the construction of the kernel in the integral (4) has remained the same as in the Temlyakov-type integral with density given on the boundary of the domain D, the proof of Theorems 1 and 2 is analogous to the proof of identical propositions ($^{5-7}$) for the Temlyakov-type integral with density given on the boundary of the domain D.

3. Consider the domains:

$$\begin{aligned} q^{+}_{s,j} &= \{w, z:\ f_1(|w|,|z|,1) < 0,\ f_3(|w|,|z|,\alpha_j) < 0,\ s = -1 \text{ or} \\ &\qquad f_2(|w|,|z|,0) > 0,\ f_3(|w|,|z|,\beta_j) < 0,\ s = +1,\ j = 1,2,\dots,n\}; \\ q^{-}_{s,j} &= \{w, z:\ f_1(|w|,|z|,\beta_j) < 0,\ f_3(|w|,|z|,0) < 0,\ s = -1 \text{ or} \\ &\qquad f_2(|w|,|z|,\alpha_j) > 0,\ f_3(|w|,|z|,1) < 0,\ s = +1,\ j = 1,2,\dots,n\}; \\ q^{(m,l)}_{s,j} &= \{w, z:\ f_1(|w|,|z|,\beta_j) \geq 0,\ f_3(|w|,|z|,\alpha_j) \geq 0,\ f_1(|w|,|z|,1) < 0,\ f_3(|w|,|z|, \end{aligned}$$

$0) < 0,\ s = -1$ or $f_2(|w|,|z|,\alpha_j) \leq 0,\ f_2(|w|,|z|,0) > 0,\ f_3(|w|,|z|,\beta_j) \geq 0,\ f_3(|w|,|z|,1) < 0,\ s = +1;\ j = 1,2,\dots,n\}$, where $f_3(|w|,|z|,\tau) = 2|z| - f_1(|w|,|z|,\tau)$.

In Fig. 1 an image of the domains is given for the case $n = 1$, with the notation $q^{+}_{-1,1} = q^{+},\ q^{-}_{-1,1} = q^{-},\ q^{(m,l)}_{-1,1} = q^{(m,l)},\ B_{-1,1} = B$.

The family of two-dimensional surfaces passing through the points of the circles $B_{s,j}$ from the domains of nonanalyticity of the integral (4) will be denoted by

$$\sigma^{(m,l)}_{s,j} = \{w, z:\ c_j^2|w|^2 + d_j^2|z|^2 + 2mc_{jd}j|w|\,|z| - 1 = 0\ (d_j > 0,\ c_j > 0,\ |m| \leq 1) \text{ and } \arg w - \arg z = l\ (\arg_s 0 = \tfrac{1}{2}[($$

Introduce the notation

$$f_q(w_0, z_0) = \lim_{\substack{(w,z)\to(w_0,z_0)\\ (w,z)\in q}} f(w,z),$$

$$f_{q^{(m,l)}}(w_0, z_0) = \lim_{\substack{(w,z)\to(w_0,z_0)\\ (w,z)\in\sigma^{(m,l)}_{s,j}\subset q^{(m,l)}_{s,j}}} f(w,z).$$

Fig. 1

Theorem 3. Let the boundary of the domain D have order n, let the density of the integral (4) be a function $\Phi(\tau, t, \xi, \eta) \in \alpha$, and let $\Phi(\tau, t + 2\pi, \xi, \eta) =$

$\Phi(\tau, t, \xi, \eta)$. Then the limiting values of the Temlyakov-type integral of the first kind (4) at points of the circles $B_{s,j}$ are determined by the formulas:

1) $f_{q^+}(w_0, z_0) = f(w_0, z_0) + f_1(w_0, z_0)$;
2) $f_{q^-}(w_0, z_0) = f(w_0, z_0) - f_1(w_0, z_0)$;
3) $f_{q^{(m,l)}}(w_0, z) = f(w_0, z_0) + f_1(w_0, z_0) - f_2(w_0, z_0)$,

where

$$f(w_0, z_0) = (4\pi^2 i)^{-1} \int_0^1 d\tau \int_0^{2\pi} dt \int_{|\xi|=1} \Phi(\tau, t, \xi, \xi e^{-it})(\xi - u_0)^{-1}\, d\xi,$$

$$f_1(w_0, z_0) = (4\pi)^{-1} \int_{\alpha_j}^{\beta_j} d\tau \int_0^{2\pi} \Phi(\tau, t, u_0, u_0 e^{-it})\, dt, \tag{6}$$

$$f_2(w_0, z_0) = (2\pi)^{-1} \int_{\alpha_j}^{\beta_j} d\tau \int_{l-\varphi_m}^{l+\varphi_m} \Phi(\tau, t, u_0, u_0 e^{-it})\, dt,$$

$$\varphi_m = \arccos m,$$

$$u_0 = c(\tau) w_0 + d(\tau) z_0 e^{it} \quad \text{for } \tau \in [0, \alpha_j) \cup (\beta_j, 1];$$

$$u_0 = \exp \frac{i}{2}\left[(1-s)\arg w_0 + (1+s)(\arg z_0 + t)\right] \quad \text{for } \tau \in [\alpha_j, \beta_j];$$

the singular integral in formula (6) is understood in the sense of the Cauchy principal value.

Proof is based on the representation of the function defined by the Temlyakov-type integral in a neighborhood of the circles $B_{-1,j}$ $(j = 1, 2, \dots, n)$, namely on the formula

$$f(w, z) = (2\pi)^{-1} \left[\int_0^{\tau_0} d\tau \int_0^{2\pi} F_{-1}(\tau, t, u, u e^{-it})\, dt + \right.$$

$$+ \int_{\tau_0}^{\tau_1} d\tau \int_{\psi+\varphi}^{2\pi+\psi-\varphi} F_{-1}(\tau, t, u, u e^{-it})\, dt + \int_{\tau_0}^{\tau_1} d\tau \int_{\psi-\varphi}^{\psi+\varphi} F_1(\tau, t, u, u e^{-it})\, dt +$$

$$\left. + \int_{\tau_1}^{1} d\tau \int_0^{2\pi} F_1(\tau, t, u, u e^{-it})\, dt \right],$$

where

$$F_{-1}(\tau,t,u,ue^{-it}) = (2\pi i)^{-1}\int_{|\zeta|=1}\Phi(\tau,t,u,ue^{-it})(\zeta-u)^{-1}\,d\zeta$$

for $|u|<1$, and the function $F_1(\tau,t,u,ue^{-it})$ is defined by the same formula, but for $|u|>1$;

$$\varphi = \arccos\left[(1-c^2(\tau)|w|^2-d^2(\tau)|z|^2)(2c(\tau)d(\tau)|w|\,|z|)^{-1}\right],$$

$$\psi = \arg w - \arg z, \qquad \tau_0 = \inf_{0\le\tau\le 1}\{\tau : f_3(|w|,|z|,\tau)=0\},$$

$$\tau_1 = \sup_{0\le\tau\le 1}\{\tau : f_1(|w|,|z|,\tau)=0\}.$$

Remark 4. Theorem 3 is formulated for the cases $a\neq 0$, $b\neq -\infty$ (or $a=0$, $b\neq-\infty$, or $a\neq 0$, $b=-\infty$, or $a=0$, $b=-\infty$), with $\alpha_1\neq 0$ and $\beta_n\neq 1$.

Remark 5. Theorem 3 for the case of domains D of type A ($D=\{w,z : c|w|+d|z|<1,\ c>0,\ d>0\}$) was considered earlier in the authors' papers ([8–10]).

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REFERENCES

1. A. A. Temlyakov, DAN, 120, No. 5 (1958).
2. A. A. Temlyakov, Izv. AN SSSR, ser. matem., 21, 89 (1957).
3. A. A. Temlyakov, Uch. zap. Mosk. obl. ped. inst. im. N. K. Krupskoi, 21, 7 (1954).
4. B. A. Fuks, *Introduction to the Theory of Analytic Functions of Several Complex Variables*, Moscow, 1962.
5. L. A. Aizenberg, DAN, 120, No. 5 (1958).

6. L. A. Aizenberg, DAN, 125, No. 5 (1959).

7. L. A. Aizenberg, Uch. zap. Mosk. obl. ped. inst. im. N. K. Krupskoi, 96, 15 (1960).

8. G. L. Lukankin, DAN, 161, No. 1 (1965).

9. G. L. Lukankin, Uch. zap. Mosk. obl. ped. inst. im. N. K. Krupskoi, 166, 49 (1966).

10. V. I. Boganov, ibid., 166, 81 (1966).

Note: Figure translations are in progress. See original paper for figures.