ON AN INHOMOGENEOUS SINGULAR CAUCHY PROBLEM

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Abstract

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MATHEMATICS

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ON AN INHOMOGENEOUS SINGULAR CAUCHY PROBLEM

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Consider, in the $(n+1)$-dimensional domain $\Omega[-\infty < x < \infty,\ 0 \le s < \infty,\ x = (x_1, \dots, x_n)]$, the solution $u(x, s; a, b, c)$ of the inhomogeneous singular Cauchy problem

$$Xu = u_{ss} + \frac{a}{s}u_s + \left(b^2 + \frac{c}{s^2}\right)u - \frac{c}{s^2}\tau(x), \quad u(x,0) = \tau(x), \quad u_s(x,0) = 0, \tag{1}$$

where $a = 2\beta \ge 0;\ b,\ c \le \nu^2;\ \nu = \beta - \frac{1}{2}$ are constants; $\tau(x) \in C^\infty$ is given on the entire hyperplane $\mathfrak{S}_n(-\infty < x_k < \infty;\ k = 1, 2, \dots, n)$; X is a linear differential operator independent of s, acting with respect to x, and $X[0] = 0$. Comparing $u(x, s)$ with $z(x, s; a, b) = u(x, s; a, b, 0)$, we arrive at the following results:

1. Let $2^{2m} m! (\nu + 1)_m g_m(\nu) = (-1)^m, \quad b_0 = \sqrt{b_1^2 - b_2^2}, \quad 4\sigma = b_0^2 s^2$. Then

$$u(x, \lambda s; a_2, b_2, c) = \sum_{m=0}^{\infty} A_m s^{2m} (X - b_1^2)^m z(x, s; a_1 + 2m, b_1), \tag{2a}$$

$$A_m = g_m(\nu_1) \sum_{n=0}^{m} \frac{(-m)_n (\nu_1 + 1)_n \lambda^{2n}}{(p_2 + 1)_n (q_2 + 1)_n}\, {}_1F_2(n+1;\ p_2+n+1,\ q_2+n+1;\ \lambda^2\sigma). \tag{2b}$$

Here $\lambda = \text{const};\ p_k$ and q_k are the roots of the equation $\rho^2 - \nu_k \rho + c/4 = 0$.

$$u(x, \lambda s; a_2, b_2, c) = \sum_{m=0}^{\infty} \bar{A}_m s^{2m} (X - b_1^2)^m z(x, s; a_1 + 4m, b_1), \tag{3a}$$

$$\bar{A}_m = \bar{g}_m(\nu_1) \sum_{n=0}^{m} \frac{(-m)_n (\nu_1 + m)_n \lambda^{2n}}{(p_2 + 1)_n (q_2 + 1)_n}\, {}_1F_2(n+1;\ p_2+n+1,\ q_2+n+1;\ \lambda^2\sigma), \tag{3b}$$

where $2^{2m} m!(\nu+m)_m \bar{g}_m(\nu) = (-1)^m$. With the aid of the equality

$$s^{2m}(X-b^2)^m z(x,s;a+4m,b) = \sum_{n=0}^{m} \gamma_n z(x,s;a+2n,b), \tag{4}$$

$$n!(\nu+1)_n \gamma_n = 2^{2m}(-m)_n(\nu+1)_{2m}(\nu+m)_n \quad (m,n=0,1,2,...),$$

one may pass from (3a) to an expansion in the family $\{z(a+2n)\}$.

For example, when $a_2 = a_1 = a,\ b_2 = b_1 = b$, substituting (4) into (3a), we find:

$$u(x,s;a,b,c) = \sum_{n=0}^{\infty} \frac{pq\Gamma(p+n)\Gamma(q+n)}{n!\,\Gamma(\nu+n+1)} z(x,s;a+2n,b). \tag{5}$$

Under the conditions $a_2 > a_1 \geq 0, \quad \beta_0 = \beta_2 - \beta_1, \quad \Gamma(\beta_0)\Gamma(\nu_1+1)\mu =$

$$= 2\Gamma(p_2+1)\Gamma(q_2+1),$$

(2), (3), and (5) generates the connection formula

$$u(x,s;a_2,b_2,c) = \int_0^1 \xi^{a_1}(1-\xi^2)^{\beta_0-1} Q(\xi,s) z(x,\xi s;a_1,b_1)\,d\xi, \tag{6a}$$

$$Q(\xi,s) = \mu\Xi_2[p_2,q_2,\beta_0;1-\xi^2,\sigma(1-\xi^2)]. \tag{6b}$$

When $b_2 = b_1 = b$, (2b) and (3b) give

$$\begin{aligned} A_m &= g_m(\nu_1)\,{}_3F_2(-m,\nu_1+1,1;\,p_2+1,q_2+1;\lambda^2), \\ \overline{A}_m &= \overline{g}_m(\nu_1)\,{}_3F_2(-m,\nu_1+m,1;\,p_2+1,q_2+1;\lambda^2), \end{aligned} \tag{7}$$

and (6b) becomes $Q_1 = \mu F(p_2,q_2;\beta_0;1-\xi^2)$. Conversely, in the case $c=0$ $(p=0,\ q=\nu)$, where $\Gamma(\beta_0)\Gamma(\nu_1+1)\mu_2 = 2\Gamma(\nu_2+1)$,

$$A_m = g_m(\nu_1)\Xi_2(-m,\nu_1+1;\nu_2+1;\lambda^2,-\lambda^2\sigma), \tag{8a}$$

$$\overline{A}_m = \overline{g}_m(\nu_1)\Xi_2(-m,\nu_1+m;\nu_2+1;\lambda^2,-\lambda^2\sigma), \tag{8b}$$

the series (6b) reduces to $Q_2 = \mu_2 \overline{I}_{\beta_0-1}(b_0 s\sqrt{1-\xi^2})$ (1). For $a_2 = a_1 = a,\ c = 0$, from (3b) and (8b) there arises the addition theorem

$$z(x,s;a,b_2) = \sum_{m=0}^{\infty} \overline{A}_m s^{2m}(X-b_1^2)^m z(x,s;a+4m,b_1), \tag{9a}$$

$$(\nu+1)_{2m}\overline{A}_m = \overline{g}_m(\nu)\sigma^m \overline{I}_{\nu+2m}(b_0,s), \tag{9b}$$

inverting which, we arrive at the equality

$$\overline{I}_\nu(b_0 s) z(x,s;a,b_1) = \sum_{n=0}^{\infty} B_n (b_0 s^2)^{2n} (X - b_1^2)^n z(x,s;a+4n,b_2), \tag{10}$$

where $(\nu+1)_{2n} 2^{2n} B_n = (-1)^n g_n(\nu)$. It is also worth noting the case $p_2 = \beta_0,\ q_2 = \nu_1\ (a_1 = 2q_2+1)$, when $A_m = g_m(q) F(-m,1,p+1;\lambda^2)$,

$$u(x,\lambda s;a,b,c) = \sum_{m=0}^{\infty} A_m s^{2m} (X - b^2)^m z[x,s;2(q+m)+1,b], \tag{11a}$$

$$u(x,s;a,b,c) = p \int_0^1 (1-\eta)^{p-1} z(x,s\sqrt{\eta};2q+1,b)\, d\eta \quad (p>0). \tag{11b}$$

2. Let us replace in (1) s,a,c by $2\sqrt{\varepsilon s}$, $2\varepsilon - 1$, $4c\varepsilon$ and pass to the limit as $\varepsilon \to \infty$. Then (1) is reduced to the Cauchy problem

$$Xv = v_s + \left(b^2 + \frac{c}{s}\right) v - \frac{c}{s}\tau(x), \qquad v(x,0) = \tau(x), \tag{12a}$$

$$v(x,s;b,c) = \lim_{\varepsilon\to\infty} u(x,2\sqrt{\varepsilon s};2\varepsilon-1,b,4c\varepsilon). \tag{12b}$$

Conversely, conflating u and z according to the rule

$$\lim_{\varepsilon\to\infty} u(x,2\sqrt{\varepsilon s};2\varepsilon-1,b,c) = \lim_{\varepsilon\to\infty} z(x,2\sqrt{\varepsilon s};2\varepsilon-1,b) = w(x,s;b), \tag{13}$$

we arrive at the problem, regular with respect to s, with initial condition

$$Xw = w_s + b^2 w, \qquad w(x,0) = \tau(x), \qquad (x,s) \in \Omega. \tag{14}$$

For $a_1 = 2\varepsilon - 1,\ a_2 = a,\ s = 2\sqrt{\varepsilon s_1},\ \lambda = \sqrt{\lambda_1/\varepsilon},\ \varepsilon \to \infty$, (7) and (8a) give

$$u(x,2\sqrt{\lambda_1 s_1};a,b,c) = \sum_{m=0}^{\infty} A_m^{(1)} s_1^m (X - b^2)^m w(x,s_1;b), \tag{15a}$$

$$z(x,2\sqrt{\lambda_1 s_1};a,b_2) = \sum_{m=0}^{\infty} \overline{A}_m^{(1)} s_1^m (X - b^2)^m w(x,s_1;b_1), \tag{15b}$$

where

$$m! A_m^{(1)} = (-1)^m {}_2F_2(-m,1;\,p+1,q+1;\lambda_1),$$

$\overline{A}_m^{(1)}$ is a Humbert function

$$\overline{A}_m^{(1)} = \frac{(-1)^m}{m!} \Phi_3(-m,\nu+1;\lambda_1,b_0^2\lambda_1 s_1). \tag{15c}$$

Putting in (2) and (3) $a_1 = a,\ a_2 = 2\varepsilon - 1,\ b_1 = b_2 = b,\ c = 4\varepsilon c_1,\ \lambda = 2\lambda_1\sqrt{\varepsilon},\ \varepsilon \to \infty$, we obtain

$$v(x, \lambda_1^2 s^2; b, c_1) = \sum_{m=0}^{\infty} A_m^{(2)} s^{2m}(X - b^2)^m z(x, s;\ a + 2m, b), \tag{16a}$$

$$v(x, \lambda_1^2 s^2; b, c_1) = \sum_{m=0}^{\infty} \overline{A}_m^{(2)} s^{2m}(X - b^2)^m z(x, s;\ a + 4m, b), \tag{16b}$$

$$A_m^{(2)} = \overline{g}_{m3} F_1(-m, \nu + 1, 1;\ c_1 + 1;\ 4\lambda_1^2),$$

$$\overline{A}_m^{(2)} = \overline{g}_{m3} F_1(-m, \nu + m, 1;\ c_1 + 1;\ 4\lambda_1^2).$$

Let us also note a relation, similar to (15b), with the Humbert function Φ_1:

$$v(x, \lambda s;\ b_2, c) = \sum_{m=0}^{\infty} A_m s^m (X - b_1^2)^m w(x, s;\ b_1), \tag{17a}$$

$$m! A_m = (-1)^m \Phi_1(1, -m;\ c + 1;\ \lambda,\ b_0^2 \lambda s). \tag{17a}$$

3. When $b_2 = b_1 = b$, (8b) reduces to Jacobi polynomials
$\overline{A}_m = \overline{g}_m(\nu_1) G_m(\nu_1, \nu_2 + 1; \lambda^2)$, and therefore, if $z(x, \lambda s;\ a_2, b) = f(\lambda)$, then

$$s^{2m}(X - b^2)^m z(x, s;\ a_1 + 4m, b) = \delta_m \int_0^1 \lambda^{\alpha_2}(1 - \lambda^2)^{-\beta_0 - 1} \overline{A}_m(\lambda) f(\lambda)\, d\lambda,$$

$$\Gamma(m - \beta_0)\Gamma(\nu_2 + 1)\overline{g}_m(\nu_1)\delta_m = (-1)^m 2^{2m+1}(\nu_2 + 1)_m \Gamma(\nu_1 + 2m + 1). \tag{18}$$

For $b_2 = b_1 = b$, it follows from (15c) that
$(\nu + 1)_m \overline{A}_m^{(1)} = (-1)^m L_m^{(\nu)}(\lambda_1)$, and the Fourier coefficients of the function $z(x, 2\sqrt{\lambda_1} s;\ a, b) = f(\lambda_1)$ are

$$s_1^m (X - b^2)^m w(x, s_1;\ b) = \frac{(-1)^m m!}{\Gamma(\nu + 1)} \int_0^\infty \lambda_1^\nu e^{-\lambda_1} L_m^{(\nu)}(\lambda_1) f(\lambda_1)\, d\lambda_1. \tag{19}$$

Replacing the bases of the expansions (15), (16b), (17a) by their integral representations (18) and (19), we arrive at analogous relations for u and v. For example, (17a) and (19) give

$$v(x,s;\,b_2,c)=\int_0^\infty \lambda^{\nu/2}R(\lambda,s)\,z(x,2\sqrt{\lambda}s;\,a,b_1)\,d\lambda, \tag{20a}$$

$$\Gamma(\nu+1)R=\int_0^\infty \xi^{\nu/2}e^{-\xi}J_\nu(2\sqrt{\lambda\xi})\Phi_2(1,c,c+1;\,b_0^2s,\xi)\,d\xi. \tag{20b}$$

Let us also note the relation, similar to (19) and (20a),

$$v(x,s;\,b,p)=\frac{1}{\Gamma(q+1)}\int_0^\infty \lambda^q e^{-\lambda}u(x,2\sqrt{\lambda}s;\,a,b,c)\,d\lambda \quad (q>-1), \tag{21}$$

and, in the case $c_2>c_1>-1$, the addition theorem:

$$v(x,s;\,b,c_2)=\frac{\Gamma(c_2+1)}{\Gamma(c_1+1)\Gamma(c_2-c_1)}\int_0^1 \xi^{c_1}(1-\xi)^{c_2-c_1-1}v(x,\xi s;\,b,c_1)\,d\xi. \tag{22}$$

If $X=\partial/\partial x_1+\dots+\partial/\partial x_n$, then
$w=e^{-b^2s}\tau(x_1+s,\dots,x_n+s)$, and here (15), (17), and (22) (for $c_1=0$) give the solutions u,z,v, while (16) (for $c_1=0$), (19), and (20) (for $c=0$) invert the corresponding resolving operators with respect to $\tau(x_1+s,\dots,x_n+s)$. Suppose, further, that $a=n-1,\ b=0,\ X$ is the Laplacian

$$X[z]=\Delta_x[z]=\sum_{i=1}^n z_{x_ix_i}.$$

Then $z(x,s;$

$n-1,0)=M[x,s;\tau(x)]$. This means that (6a) in the case $a_1=n-1,\ b_1=0$, $a_2>n-1,\ X=\Delta_x$, and also (20a) for $a=n-1,\ b_1=0$, give solutions u and v of problems (1), (12a). For example, here (6), when $n=1$, take the form

$$u(x,s;a,b,c)=\int_{-1}^1 (1-\xi^2)^{\beta-1}Q_0(\xi,s)\tau(x+\xi s)\,d\xi, \tag{23a}$$

$$Q_0(\xi,s)=\mu_0\Xi_2[p,q,\beta;1-\xi^2,-\sigma_0(1-\xi^2)], \tag{23b}$$

where $\sqrt{\pi}\Gamma(\beta)\mu_0=\Gamma(p+1)\Gamma(q+1)$, $4\sigma_0=b^2s^2$, and from (20a) it follows that

$$v(x,s;0,c)=\frac{\Gamma(c+1)}{\sqrt{\pi}}\int_{-\infty}^\infty e^{-\xi}\Psi(c,1/2;\xi^2)\tau(x+2\xi\sqrt{s})\,d\xi. \tag{24}$$

Analogous assumptions reduce (2a), (3a), (8), and (15b) to expansions of the means $M(x,s;\tau)$ in the basic series over the systems $\{z(a+2m)\}$, $\{z(a+4m)\}$, $\{(X-b^2)^m w\}$.

Finally, let $b=0$, $Xz=\mathfrak{B}_x[z]=z_{xx}+\dfrac{2a}{x}z_x$. Then

$$z=\gamma\int_{-1}^{1}\tau(x+\xi s)(1-\xi^2)^{\beta-1}(1+\xi t)^{\alpha}F(\alpha,1-\alpha,\beta;\omega)\,d\xi, \qquad (25)$$

$$\sqrt{\pi}\Gamma(\beta)\gamma=\Gamma(\nu+1), \qquad t=-s/x, \qquad 4(1+\xi t)\omega=t^2(\xi^2-1).$$

Substituting (25) into (6a) ($b_1=0$), (11b), (19), and (20a) ($b=0$), we obtain u,v,w for $X=\mathfrak{B}_x$.

4. To power data $\tau(x)=x^{\alpha}$ ($\alpha=\text{const}$) for $X=\partial/\partial x$ there correspond the solutions

$$u_1=x^{\alpha}\,{}_2F_2(-\alpha,1;p+1,q+1;-t_1), \qquad w_1=e^{-\sigma_1}(x+s)^{\alpha},$$

$$z_1=x^{\alpha}\Phi_3(-\alpha,\nu+1;-t_1,-\sigma_0), \qquad v_1=x^{\alpha}\Phi_1(-\alpha,1,c+1;-t,-\sigma_1),$$

where $t_1=s^2/4x$, $\sigma_1=b^2s$. If, however, $X=\partial^2/\partial x^2$, then

$$z_2=x^{\alpha}\Xi_2\left(-\frac{\alpha}{2},\frac{1-\alpha}{2},\nu+1;t^2,-\sigma_0\right),$$

$$u_2=x^{\alpha}\,{}_3F_2\left(-\frac{\alpha}{2},\frac{1-\alpha}{2},1,p+1,q+1;t^2\right).$$

Finally, confluencing z_2 and u_2 according to the rules (12b), (13), we obtain

$$w_2=x^{\alpha}e^{-\sigma_1}\times{}_2F_0\left(-\frac{\alpha}{2},\frac{1-\alpha}{2};t_2\right),$$

$$v_2=x^{\alpha}\,{}_3F_1\left(-\frac{\alpha}{2},\frac{1-\alpha}{2},1,c+1;t_2\right),$$

where $t_2=4s/x^2$.

In the case $\mathfrak{B}_x z=z_{ss}+\dfrac{a}{s}z_s$, the solutions z and $\overline{z}$ of Tricomi' s problems from (2) are written in quadratures with the aid of the Green-Hadamard functions $H(x,s;x_0,s_0)$, $\overline{H}(x,s;x_0,s_0)$, which here have the form:

$$H = \chi t^{1-a}\left(\frac{2s_0}{R}\right)^a\left(\frac{x}{x_0}\right)^\alpha H_2(1-\beta, 1-\beta, \alpha, 1-\alpha, 2-a; t, \rho),$$

$$\overline{H} = \overline{\chi}\left(\frac{2s_0}{R}\right)^a\left(\frac{x}{x_0}\right)^\alpha H_2(\beta, \beta, \alpha, 1-\alpha, a; t, \rho),$$

where $tR^2 = 4ss_0$, $4xx_0\rho = R^2$, $R = \sqrt{(x-x_0)^2-(s-s_0)^2}$; χ and $\overline{\chi}$ are indicated in (2).

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Note: Figure translations are in progress. See original paper for figures.