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Abstract

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ON ONE METHOD OF DESCRIBING THE VISCOELASTIC BEHAVIOR OF POLYMERIC BODIES

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In describing the viscoelastic behavior of polymeric bodies, integral-operator relations of Volterra—Volterra type have become widely used; moreover, fractionally exponential functions, introduced by Yu. N. Rabotnov ([1]), are often used as kernels of the time operators. For example, in ([2,3]) it is noted that, by a suitable choice of the exponents of fractional order, it is possible to approximate satisfactorily the experimental creep curves of certain reinforced plastics.

As shown in ([1]), such an approach is convenient from the mathematical point of view. It makes it possible to construct elements of an operator theory that includes the simplest operations on them, which makes it possible to realize a fairly broad class of functions of operators. The latter circumstance is especially important in solving boundary-value problems of linear viscoelasticity within the framework of Volterra' s principle.

It turns out that one can construct an analogue of Yu. N. Rabotnov' s theory for a broader class of kernels.

1. Let us consider a Volterra operator of the form

$$K_\alpha^*(\beta)(...) = \int_0^t K_\alpha(\beta, t-\tau)(...)\,d\tau, \tag{1}$$

where

$$|K_\alpha^*(\beta)| = K_\alpha(\beta, t) = t^{\alpha-1}e^{-\beta t}/\Gamma(\alpha), \quad 0 < \alpha < 1,\ \beta \geq 0; \tag{2}$$

$\Gamma(\alpha)$ is Euler' s gamma function. Here we also note that the use of kernels of type (2) as influence functions can, to some extent, be physically justified. For this, as is known ([4]), it is necessary to show that the functions (2) can be put into correspondence with some nonnegative retardation spectrum. It is easy to verify that such a spectrum is the interval $(0, 1/\beta)$. In ([5,6]) it is noted that kernels (2) describe well experimental data on the creep of various materials, including glass-reinforced plastics. In this connection, the construction of a phenomenological theory based on the kernel (2) is not devoid of meaning.

Let us first show that the powers of operator (1) possess the same properties as the powers of the Abel operator ([1]). Indeed, for the kernel of the square of operator (1) we have

$$|K_\alpha^{*2}(\beta)| = \frac{e^{-\beta(t-\tau)}}{\Gamma^2(\alpha)} \int_\tau^t (t-s)^{\alpha-1}(s-\tau)^{\alpha-1}\, ds = \frac{(t-\tau)^{2\alpha-1}e^{-\beta(t-\tau)}}{\Gamma(2\alpha)}.$$

This means that

$$K_\alpha^{*2}(\beta) = K_{2\alpha}^*(\beta). \tag{3}$$

By the method of induction one can show that

$$K_\alpha^{*\,n}(\beta) = K_{n\alpha}^*(\beta). \tag{4}$$

2. Let us introduce the operator $1 - \varkappa K_\alpha^*(\beta)$, where $\varkappa$ is a number. We find its inverse operator, which we denote by

$$[1 - \varkappa K_\alpha^*(\beta)]^{-1} = 1 + \varkappa E_\alpha^*(\beta, \varkappa). \tag{5}$$

From the definition (1) of the operator $K_\alpha^*(\beta)$ there follow (7) the existence and uniqueness of the inverse operator (5). For its actual computation we use Volterra’ s symbolic method ([1]). We find

$$E_\alpha^*(\beta, \varkappa) = \sum_{n=1}^{\infty} \varkappa^{n-1} K_{n\alpha}^*(\beta). \tag{6}$$

Here the property (4) of powers of the operator $K_\alpha^*(\beta)$ has been used. If we recall that

$$|K_{n\alpha}^*(\beta)| = t^{n\alpha-1}e^{-\beta t}/\Gamma(n\alpha),$$

then for the kernel of the operator (6) we obtain the expression

$$E_\alpha(\beta, \varkappa, t) = e^{-\beta t} \sum_{n=1}^{\infty} \frac{\varkappa^{n-1} t^{n\alpha-1}}{\Gamma(n\alpha)} = e^{-\beta t} \mathcal{E}_\alpha(\varkappa, t), \tag{7}$$

where $\mathcal{E}_\alpha(\varkappa, t)$ is the exponential of fractional order. We note that the resolvent (7) of the kernel (2) by operational-calculus methods was also found independently in work ([6]).

3. From the definition of the inverse operator (5) it follows that

$$[1-\varkappa K_\alpha^*(\beta)]\,[1+\varkappa E_\alpha^*(\beta,\varkappa)]=1$$

or

$$E_\alpha^*(\beta,\varkappa)K_\alpha^*(\beta)=\frac{1}{\varkappa}\,[E_\alpha^*(\beta,\varkappa)-K_\alpha^*(\beta)]. \tag{8}$$

Taking into account that $K_\alpha^*(\beta)=E_\alpha^*(\beta,0)$, equality (8) may be regarded as an analogue of the multiplication theorem ([1]) for $\mathcal{E}$-operators, for E-operators with a zero value of the parameter in one factor. However, it is not difficult to verify that the multiplication theorem for E-operators holds for arbitrary values of the parameters. Indeed, from the definition of the product of operators and from the properties of the kernel (7) we have:

$$|E_\alpha^*(\beta,\varkappa)E_\alpha^*(\beta,\mu)|=e^{-\beta(t-\tau)}\,|\mathcal{E}_\alpha^*(\varkappa)\mathcal{E}_\alpha^*(\mu)|\,. \tag{9}$$

But from the multiplication theorem for $\mathcal{E}$-operators ([1]) it follows that

$$|\mathcal{E}_\alpha^*(\varkappa)\mathcal{E}_\alpha^*(\mu)|=\frac{1}{\varkappa-\mu}[\mathcal{E}_\alpha(\varkappa,t)-\mathcal{E}_\alpha(\mu,t)].$$

Using this in (9), with account of (7), we obtain

$$|E_\alpha^*(\beta,\varkappa)E_\alpha^*(\beta,\mu)|=\frac{1}{\varkappa-\mu}[E_\alpha(\beta,\varkappa,t)-E_\alpha(\beta,\mu,t)]. \tag{10}$$

This means that

$$E_\alpha^*(\beta,\varkappa)E_\alpha^*(\beta,\mu)=\frac{1}{\varkappa-\mu}[E_\alpha^*(\beta,\varkappa)-E_\alpha^*(\beta,\mu)]. \tag{11}$$

i.e., for E-operators the multiplication theorem holds. In the case $\varkappa=\mu$,

$$E_\alpha^{*2}(\beta,\varkappa)=\partial E_\alpha^*(\beta,\varkappa)/\partial\varkappa, \tag{12}$$

where by the derivative of an operator with respect to a parameter is meant the new operator whose kernel is obtained by differentiating, with respect to the same parameter, the kernel of the original operator.

A consequence of the multiplication theorem (11) is the inversion theorem for E-operators

$$[1 - \mu E^*_\alpha(\beta, \varkappa)]^{-1} = 1 + \mu E^*_\alpha(\beta, \varkappa + \mu). \tag{13}$$

From the multiplication theorem (11) and the inversion theorem (13) it follows that rational operations on E-operators lead to operators of the same class. Mechanically this means that, if functions of type (7) are chosen as creep kernels, then the relaxation kernels will be functions of the same type.

4. Volterra' s principle, together with elements of operator theory, provides broad possibilities in solving boundary-value problems for polymer bodies. In many practically important problems the boundary conditions do not contain time (creep, relaxation). All calculations in the solution of such problems reduce to tabulating certain functions that are the result of the action of the corresponding operators on unity. For example:

$$E^*_\alpha(\beta, \varkappa)(1) = \frac{1}{\varkappa} \sum_{n=1}^{\infty} \left(\frac{\varkappa}{\beta^\alpha}\right)^n \frac{\gamma(n\alpha, t/\beta)}{\Gamma(n\alpha)}, \tag{14}$$

where $\gamma(n\alpha, t/\beta)$ is the incomplete gamma function (8).

Sometimes it is not necessary to study the entire creep or relaxation process; it is sufficient to evaluate the behavior at large times. In this case the limiting values for the functions mentioned will be useful. Since

$$\gamma(n\alpha, t/\beta) \to \Gamma(n\alpha) \quad \text{as} \quad t \to \infty,$$

relation (14) gives

$$E^*_\alpha(\beta, \varkappa)(1) \to \begin{cases} 1/(\beta^\alpha - \varkappa), & \text{if } |\varkappa/\beta^\alpha| < 1, \\ \text{does not exist}, & \text{if } |\varkappa/\beta^\alpha| \geq 1. \end{cases} \tag{15}$$

Conditions (15) are a criterion of stability or instability in time of the behavior of a viscoelastic body. For example, if $E_\alpha(\beta, \varkappa, t)$ is the creep kernel, then when the first condition in (15) is satisfied, bounded creep will occur, i.e., the creep process is stable in time; when the second condition is satisfied, the creep process is unbounded.

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Note: Figure translations are in progress. See original paper for figures.