ON AN ANALOGUE OF THE ANDRONOV-WITT THEOREM

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Abstract

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MATHEMATICS

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ON AN ANALOGUE OF THE ANDRONOV-WITT THEOREM

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1°. In this paper sufficient conditions are obtained for the asymptotic orbital stability of a completely bounded solution of an autonomous system of ordinary differential equations, analogous to the well-known Andronov-Witt theorem ([1]) on the stability of a periodic solution.

2°. Consider the real linear system

$$dx/dt = A(t)x, \tag{1}$$

where

$$\mathbf{x} = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} \equiv \operatorname{colon}(x_1, \dots, x_n),$$

$$A(t) \in C(I_t^+) \quad (I_t^+ = (t_0, +\infty)) \text{ is an } n \times n\text{-matrix;} \quad \sup_t \|A(t)\| < \infty.$$

By

$$\chi[\mathbf{x}(t)] = \lim_{t\to\infty} \left\{ \frac{1}{t} \ln \|\mathbf{x}(t)\| \right\}$$

we shall denote the **characteristic exponent** of the solution $\mathbf{x}(t)$.

Definition 1. A bounded solution $\mathbf{x}(t)$ of system (1) will be called **standard** if

$$0 < \inf_t \|\mathbf{x}(t)\| \leq \sup_t \|\mathbf{x}(t)\| < \infty.$$

Definition 2. A bounded solution $\mathbf{x}(t)$ of system (1) will be called **completely bounded** if: 1) it is standard and 2) the adjoint system

$$d\vec{\xi}/dt = A^T(t)\vec{\xi} \tag{2}$$

($A^T(t)$ is the transposed matrix of $A(t)$) has a bounded solution $\vec{\xi}(t)$, with

$$\mathbf{x}^T(t)\vec{\xi}(t) = \text{const} \neq 0.$$

Definition 3. A square matrix $L(t)$ will be called a **Lyapunov matrix** on the set $Z \subset I_t^+$ if, for $t \in Z$, the following conditions are satisfied:
a) $L(t) \in C^{(1)}$;
b) $\sup_t \|L(t)\| < \infty, \quad \sup_t \|\dot{L}(t)\| < \infty$;
c) $\inf_t |\det L(t)| > 0$.

Lemma. Let system (1) be regular in the sense of Lyapunov ([2]), and let its normal fundamental matrix $X(t) = (x_{ij}(t))$ consist of solutions

$$\mathbf{x}^{(j)}(t) = \text{colon}[x_{1j}(t), \dots, x_{nj}(t)] \quad (j = 1, \dots, n),$$

of which one, $\mathbf{x}^{(1)}(t)$, is completely bounded, while all the remaining $\mathbf{x}^{(j)}(t)$ $(j > 1)$ have negative characteristic exponents.

Then the half-plane $\{t_0 \leq t < +\infty,\ t_0 \leq \tau < +\infty\}$ can be represented as a finite sum of open sets $O_{pq} = \{t, \tau : t \in O_p,\ \tau \in O_q\}$, on each of which the Cauchy matrix

$$K(t, \tau) = X(t)X^{-1}(\tau)$$

admits the representation

$$K(t, \tau) = U_p(t)\,\text{diag}[E_1, Y_{pq}(t, \tau)]V_q^T(\tau), \tag{3}$$

where E_1 is the identity matrix of order 1, $U_p(t)$ and $V_q(\tau)$ are Lyapunov matrices on O_p and O_q, respectively, $Y_{pq}(t, \tau) \in C^{(1)}(O_{pq})$, and

$$\|Y_{pq}(t, \tau)\| \leq ce^{-\alpha(t-\tau)}e^{\varepsilon t} \tag{4}$$

for $t_0 \leq \tau \leq t$, where $\varepsilon > 0$ is arbitrary,

$$-\alpha = \max_{j>1} \chi[\mathbf{x}^{(j)}(t)] < 0$$

and $c = c(\varepsilon, t_0)$ is a positive constant.

3°. Let the real autonomous system

$$dy/dt = \mathbf{f}(\mathbf{y}), \tag{5}$$

where $\mathbf{f}(\mathbf{y}) \in C^{(2)}(R^n)$, $n \geq 2$, admit a solution $\eta = \eta(t)$ bounded on $I_t = (-\infty, +\infty)$.

Definition 4. The solution $\eta(t)$ is called **orbitally stable** as $t \to \infty$ if, for some t_0, for every $\varepsilon > 0$ there exists $\delta > 0$ such that, for every solution $\mathbf{y}(t)$ $(t_0 \leq t < \infty)$ satisfying the condition

$$\|\mathbf{y}(t_0) - \eta(t_0)\| < \delta,$$

the inequality

$$\rho(\mathbf{y}(t), L_\eta^+) = \inf_{t_0 \leq t_1 < \infty} \|\mathbf{y}(t) - \eta(t_1)\| < \varepsilon$$

will be ensured for $t_0 \leq t < \infty$, where L_η^+ is the positive semitrajectory of the solution $\eta(t)$.

If, moreover, for $\|\mathbf{y}(t_0) - \eta(t_0)\| < \Delta$, where $\Delta > 0$ is sufficiently small, we have $\rho(\mathbf{y}(t), L_\eta^+) \to 0$ as $t \to \infty$, then the solution $\eta(t)$ is called **asymptotically orbitally stable** as $t \to \infty$.

Definition 5. We shall say that the solution $\eta(t)$ has an **asymptotic phase** ([3]) if, for every solution $\mathbf{y}(t)$ sufficiently close to it at $t = t_0$, there exists a constant number c (the phase) such that

$$\lim_{t \to \infty} \|\mathbf{y}(t + c) - \eta(t)\| = 0.$$

Obviously, an orbitally stable solution that has an asymptotic phase is asymptotically orbitally stable.

Main theorem. Suppose the autonomous system (5) admits a solution $\eta(t)$, bounded on I_t, such that

$$\inf_t \|\dot{\eta}(t)\| > 0,$$

and suppose that the variational equation for this solution,

$$dx/dt = \mathbf{f}'(\eta(t))x \tag{6}$$

form a regular linear system, among whose characteristic exponents only one is zero, corresponding to the completely unbounded solution $\mathbf{x}_0 = \vec{\eta}(t)$, while the others are negative. Then the solution $\vec{\eta}(t)$ is asymptotically orbitally stable as $t \to \infty$ and has an asymptotic phase.

To prove the main theorem, by means of the change of variables

$$\mathbf{y} = \vec{\eta}(t) + \mathbf{z}$$

we bring system (5) to the form

$$dz/dt = f'(\vec{\eta}(t))z + \vec{\varphi}(t, z), \tag{7}$$

where

$$\|\vec{\varphi}(t, \tilde{z}) - \vec{\varphi}(t, z)\| \leq$$

$$\leq N \max(\|z\|, \|\tilde{z}\|)\|\tilde{z} - z\|$$

for $\|z\|, \|\tilde{z}\| < h < \infty$ (N is a constant). To find a family of solutions $z(t, a)$ of equation (7) such that $z(t, a) \to 0$ as $t \to \infty$, where $a = \mathrm{colon}(0, a_2, \dots, a_n)$ is a vector parameter, we use the singular integral equation

$$z(t, a) = X(t)a + \int_{t_0}^{\infty} G(t, \tau)\vec{\varphi}(\tau, z(\tau, a))\, d\tau \quad (t \geq t_0), \tag{8}$$

where $X(t)$ is a suitable fundamental matrix of the linear system (6) and

$$G(t, \tau) = \begin{cases} X(t)AX^{-1}(\tau), & \tau > t \geq t_0, \\ X(t)BX^{-1}(\tau), & t_0 \leq \tau < t; \end{cases}$$

$$A = -\,\mathrm{diag}(E_1, 0), \qquad B = \mathrm{diag}(0, E_{n-1})$$

(E_m is the identity matrix of order m). The integral equation (8) is solved by the method of successive approximations analogously to (4).

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CITED LITERATURE

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Note: Figure translations are in progress. See original paper for figures.