ON ELLIPTIC EQUATIONS IN UNBOUNDED DOMAINS

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Abstract

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MATHEMATICS

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ON ELLIPTIC EQUATIONS IN UNBOUNDED DOMAINS

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It is known that, in solving the exterior Dirichlet problem for the Laplace equation in a space of dimension greater than two, one must prescribe the limit of the solution at infinity. A number of works are devoted to finding conditions on the coefficients of an elliptic equation under which the solution of the exterior Dirichlet problem exists and is unique in the class of bounded functions, or in the class of functions tending to zero at infinity ([1], [2]). It turns out that a second-order equation, generally speaking, may have a nontrivial boundary at infinity, i.e., instead of the condition $\lim_{|x|\to\infty} u(x) = c$ one can prescribe different limits for different ways in which x tends to infinity. In the present note the infinitely distant boundary for a second-order operator is studied. Under certain conditions a complete description is given of the set of bounded solutions of an elliptic equation in the whole space or in an unbounded domain, i.e., a part of the Martin boundary (see ([4], [5])) corresponding to bounded functions is constructed. These results are closely connected with the question of stabilization of solutions of the Cauchy problem for parabolic equations.

Let an elliptic differential operator be given in the space R^n:

$$L_\alpha = \alpha \sum_{i,j=1}^{n} \frac{\partial}{\partial x_i}\left(a_{ij}(x)\frac{\partial}{\partial x_j}\right) + \sum_{i=1}^{n} b_i(x)\frac{\partial}{\partial x_i}.$$

The coefficients of the operator L_α are assumed bounded in the whole space together with their first derivatives; the matrix $\{a_{ij}(x)\}$ is uniformly nondegenerate and positive definite, $\alpha > 0$. We denote by x_t^a the solution of the system of ordinary differential equations

$$d(x_t^a)^i/dt = b_i(x), \qquad (x_0^a)^i = a_i, \quad i = 1, \dots, n, \quad a = \{a_1, \dots, a_n\}. \qquad (1)$$

By $\rho(A, B)$ we denote the distance between the sets A and B, $V_\varepsilon(A) = \{x : \rho(x, A) < \varepsilon\}$. In what follows we shall sometimes assume that, for some $N, T, \varepsilon, \lambda, \beta > 0$, the following conditions are satisfied.

A. At least one of the assumptions holds.

1. $\rho(x_T^a, 0) - \rho(a, 0) > 1$ for $\rho(a, 0) > N$.
2. There exists a straight line l such that either $(x_T^x)_l - x_l > 1$ (< 1) for $|x_l| > N$, or $(x_T^x)_l - x_l > 1$ for $x_l > N$ and $(x_T^x)_l - x_l < -1$ for $x_l < -N$. By x_l we denote the projection of the vector x onto the straight line l.

B. One can specify m integral curves $\gamma_k(t)$, $k = 1, \dots, m$, of system (1), for which

B$_1$. There exists an $(n-1)$-dimensional manifold Δ such that for every

$$x \in V_\varepsilon(\Delta) \cup \{x : \rho(x, 0) < N\}$$

there is a curve $\gamma_{k(x)}(t)$ such that

$$\lim_{t\to\infty} \rho(x_t^x, \gamma_{k(x)}(t)) = 0,$$

and moreover

$$\rho(x_T^x, \gamma_{k(x)}) - \rho(x, \gamma_{k(x)}) > 1$$

(by γ_k without the argument t we denote the entire curve $\gamma_k(t)$). Those x for which

$x_t^x \to \gamma_k$, denote it by Γ_k. It is assumed that $\Gamma_k \supset \{x : \rho(x, \gamma_k(t)) < \lambda t - N\}$; the sets $A_k = \Gamma_k \cap \{|x| > N\}$, for different k, are separated by the manifold Δ.

B_2. Each γ_k can be covered by a coordinate system $Y = (y_1, \dots, y_n)$ such that: 1) the curve γ_k is given by the equation $y_2 = y_3 = \dots = y_n = 0$; 2) for $\rho(y, \gamma_k) < \lambda y_1 - N$ the inequalities $b_1^Y(y) > \beta$, $b_\perp^Y(y) > \beta\rho(y, \gamma_k)$ hold, where $b_\perp^Y(y)$ is the projection of the vector $b^Y(y)$ onto the perpendicular from y to γ_k, and $b^Y(y)$ is the vector formed from the coefficients of the first derivatives in the system Y.

C. For $x \in \overline{V_\varepsilon(\Delta)} \cup V_\varepsilon\left(\bigcup_1^m \gamma_k\right) \cup \{x : \rho(x, 0) < N\}$ the inequality

$$\rho(x\tau^x, \Delta) - \rho(x, \Delta) > 1$$

is satisfied. (It is enough to assume that this inequality is satisfied only for $x \in \{x : \rho(x, \Delta) < \lambda\rho(0, x) - N\}$.)

Theorem 1. *Suppose that conditions A, B_1, C are satisfied. Then there exists $0 < \alpha_1 \le \infty$ such that, for $\alpha < \alpha_1$, there exists a unique bounded solution of the problem*

$$L_\alpha u(x) = 0, \qquad \lim_{t\to\infty} u(\gamma_k(t)) = a_k, \qquad k = 1, \dots, m, \tag{2}$$

for arbitrary $a_1, \dots, a_m$.

For the proof, consider the Markov process $X = \{x_t, p_x\}$ governed by the operator L_α (see (3)). It follows from assumption A that, for α less than some α_1^1, the process X is transient (for definitions see (2)). From conditions C and A it follows that the trajectories of the process X will spend, in $V_\varepsilon(\Delta)$, only a finite time with probability 1; hence, by assumption B_1, for $\alpha < \alpha_1^2$, for almost all trajectories with respect to the measure P_x ($x \in R^n$) there is a unique curve $\gamma_{k(\omega)}$ (its own for each trajectory) such that the trajectory $x_t(\omega)$ visits every neighborhood of $\gamma_{k(\omega)}$ after arbitrarily large times. Consider the function $u(x) = M_x a_{k(\omega)}$, where $k(\omega)$ is the number of the curve γ corresponding to the trajectory $x_t(\omega)$. It is easy to verify that the function $u(x)$ satisfies the equation $Au(x) = 0$, where A is the infinitesimal operator of the process X. From assumptions A, B_1, C it is concluded that the function $u(x)$ is continuous, and then it is proved that every continuous solution of the equation $Au = 0$ is twice continuously differentiable and $L_\alpha u(x) = 0$. From assumptions A and B_1 it is not difficult to derive that the function $u(x)$ assumes the boundary values.

To prove uniqueness, it suffices to show that every bounded solution $v(x)$ having zero limit along each curve γ_k, $k = 1, 2, \dots, m$, is identically equal to zero. Since the operator L is uniformly elliptic in R^n, and the coefficients together with their derivatives are bounded, the function $v(x)$ is uniformly continuous in R^n. This follows from the existence of an a priori estimate for the first derivatives (see (7)). Hence, for every $\delta > 0$ there exist N, $\varepsilon > 0$ such that $|v(x)| < \delta$ for

$$x \in A_\delta = \left\{ x : |x| > N, \quad x \in V_\varepsilon \left(\bigcup_1^m \gamma_k \right) \right\}.$$

Denote by τ_δ the first moment at which the set A_δ is reached. For the function $v(x)$ the equality

$$v(x) = M_x v(x_{\tau_\delta})$$

is valid. Hence, taking into account that $\sup_{x \in A_\delta} |v(x)| \le \delta$, we conclude that $|v(x)| \le \delta$. In view of the arbitrariness of δ, $v(x) \equiv 0$.

Remark 1. It is not difficult to give examples showing that the assertion of Theorem 1, generally speaking, does not hold for sufficiently large α. One can, for example, arrange that the equation $L_\alpha u = 0$ for $\alpha > \alpha_2$ would have only constants as its solutions. For some

* For α_1 one can give a lower estimate in terms of the coefficients of the operator and their derivatives.

under the assumptions $a_1 = \infty$. Such a case is considered in the author' s paper (8). There we also abandon the assumption that the number of curves γ_k is finite (the infinitely remote boundary may even have an open component). These improvements are achieved at the cost of less precise results and of narrowing the class of admissible equations.

Remark 2. If the equation $Lu = 0$ is considered not in the whole space, but in an unbounded domain D, then Theorem 1 requires small changes. If $R^n \setminus D$ is a bounded set, the changes are obvious. If $R^n \setminus D$ extends to infinity, then it is necessary to assume additionally that every curve γ_k lying in D belongs to D, together with a neighborhood of $\gamma_k(t)$ that expands linearly as the point recedes from the origin.

Remark 3. One can prove the existence and uniqueness of a bounded solution of the problem

$$L_\alpha u(x) = f(x), \qquad \lim_{t\to\infty} u(\gamma_k(t)) = a_k$$

for any function $f(x)$ decreasing sufficiently rapidly as $|x|$ grows.

Theorem 2. *Suppose that conditions A, B, C are fulfilled. Then there exists $0 < a_2 \leqslant \infty$ such that every bounded solution of the equation $L_\alpha u = 0$ for $\alpha < a_2$ has a limit along each curve γ_k, $k = 1, \dots, m$.*

It follows from Theorems 1 and 2 that, when conditions A, B, C are fulfilled and $\alpha < \min(a_1, a_2)$, the set of solutions of the equation $L_\alpha u(x) = 0$ is an m-dimensional space. A basis in this space may be given by solutions $u_1(x), \dots, u_m(x)$ with conditions

$$\lim_{t\to\infty} u_k(\gamma_j(t)) = \delta_j^k.$$

Thus we have constructed a part of the Martin boundary corresponding to bounded solutions; the functions $u_k(x)$ are minimal (see (4,5)).

We outline the proof of Theorem 2. Let $L_\alpha u(x) = 0$ and $|u(x)| < K < \infty$ for $x \in R^n$. For α sufficiently small, as was explained in the proof of Theorem 1, the trajectories of the process X, with probability 1, starting from arbitrary x, however far away, intersect one and only one curve γ_k (its own for each trajectory). Let $a > 0$ and a natural number n be chosen so that for $n > n_0$

$$P_x\{\rho(x_{\tau_n}, \gamma_k) < a\} > 1/a, \tag{3}$$

where τ_n is the moment of first reaching the set $y_1 = n$ (see condition B_2). Such a choice is possible by condition B_2.

Suppose that

$$\underline{\lim}_{t\to\infty} u(\gamma_k(t)) = a, \qquad \overline{\lim}_{t\to\infty} u(\gamma_k(t)) = b, \quad b - a = \varkappa > 0.$$

Let

$$\Gamma_+ = \{x : x = \gamma_k(t),\ |u(x) - b| < \varkappa/8\}, \quad \Gamma_- = \{x : x = \gamma_k(t),\ |u(x) - a| < \varkappa/8\},$$

$$\tilde\Gamma_+ = V_\varepsilon(\Gamma_+), \qquad \tilde\Gamma_- = V_\varepsilon(\Gamma_-).$$

Since the function $u(x)$ is uniformly continuous, by choosing $\varepsilon > 0$ sufficiently small we obtain

$$\inf_{x\in\tilde\Gamma+} u(x) > b - \varkappa/4, \qquad \sup_{x\in\tilde\Gamma-} u(x) < a + \varkappa/4.$$

From inequality (3) one can conclude that, with probability arbitrarily close to 1, for $x \in \gamma_k$ and $|x|$ sufficiently large, the trajectories visit both $\tilde{\Gamma}_+$ and $\tilde{\Gamma}_-$, which contradicts the existence of the limit

$$\lim_{t\to\infty} u(x_t(\omega)).$$

The latter limit must exist with probability 1, since $u(x_t)$ is a bounded martingale (see (6)). Theorem 2 is proved.

We now consider the Cauchy problem

$$\partial w/\partial t = L_\alpha w(t,x), \qquad w(0,x) = f(x). \tag{4}$$

It is known that, if the process X is recurrent (see (2)), then the solution of problem (4) stabilizes as $t \to \infty$ to the constant equal to

$$\int_{R^n} f(x)\mu(x)\,dx,$$

where $\mu(x)$ is the density of the invariant measure of the process X. The function $\mu(x)$

can be found as a solution of the equation $L_\alpha^* \mu(x) = 0$. In our case the process is nonrecurrent, and stabilization will not occur for every initial function $f(x)$. If stabilization does occur, then the limiting function, generally speaking, is different from a constant.

Theorem 3. *Let the conditions of Theorem 2 be satisfied and let $\alpha < \alpha_1$. Suppose that in a neighborhood of each curve γ_k (see condition B_2) there exists the limit*

$$\lim_{y_1\to\infty} \bar{f}(y_1,\dots,y_n) = \bar{f}_k(y_2,\dots,y_n)$$

and at least one of the following conditions is satisfied:

$$1) \quad \bar{f}_k(y_2,\dots,y_n) = c_k = \text{const};$$

$$2) \quad a_{ij}^{Y_k}(y_1,\dots,y_n) \Rightarrow \bar{a}_{ij}^k(y_2,\dots,y_n), \quad b_i^{Y_k}(y_1,\dots,y_n) \Rightarrow \bar{b}_i^k(y_2,\dots,y_n)$$

as $y_1 \to \infty$, $\bar{a}_{ij}^k(y)$, $\bar{b}_i^k(y) \in C^2(R^{n-1})$. Here

$$g(y_1,\dots,y_n) \Rightarrow \bar{g}(y_2,\dots,y_n)$$

means that

$$\lim_{y_1\to\infty} g(y) = \bar{g}(y_2,\dots,y_n)$$

uniformly in $y_2,\dots,y_n$, and

$$\int_{y_1^0}^{\infty} |g(y) - \bar{g}(y)|\,dy_1$$

converges uniformly in $y_2, \dots, y_n$.

Then

$$\lim_{t\to\infty} w(t,x) = u(x)$$

exists. The function $u(x)$ *can be found as the solution of problem (2), where* $a_k = c_k$ *if condition 1) is fulfilled, and*

$$a_k = \int_{R^{n-1}} \bar{f}_k(y)\mu_k(y)\,dy,$$

where $\mu_k(y)$ *is the unique positive solution of the equation*

$$\alpha \sum_{i,j=2}^{n} \frac{\partial}{\partial y_i}\left(\bar{a}_{ij}^{k}(y)\frac{\partial \mu_k}{\partial y_j}\right) - \sum_{i=2}^{n} \frac{\partial}{\partial y_i}\left(\bar{b}_i^{k}(y)\mu_k(y)\right) = 0$$

with the condition

$$\int_{R^{n-1}} \mu_k(y)\,dy = 1.$$

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Note: Figure translations are in progress. See original paper for figures.