ON A MULTIPOINT BOUNDARY-VALUE PROBLEM

MATHEMATICS

1967

SovietRxiv

Abstract

Full Text

UDC 517.941

MATHEMATICS

G. S. ZAĬTSEVA

ON A MULTIPOINT BOUNDARY-VALUE PROBLEM

(Presented by Academician A. Yu. Ishlinskii, 13 XII 1966)

Consider the equation

$$x^{(n)} + p_1(t)x^{(n-1)} + \dots + p_n(t)x = 0 \qquad (a \le t \le b), \tag{1}$$

where $p_1(t), \dots, p_n(t)$ are real continuous functions on the interval $[a, b]$,

$$|p_i(t)| \le L_i, \qquad i = 1, 2, \dots, n \qquad (a \le t \le b),$$

and the multipoint boundary-value problem

$$x(a_i) = A_{i,1}, \quad x'(a_i) = A_{i,2}, \dots, x^{(r_i-1)}(a_i) = A_{i,r_i},$$

$$i = 1, 2, \dots, m \quad (2 \le m \le n,\ r_1 + r_2 + \dots + r_m = n), \tag{2}$$

$$a \le a_1 < a_2 < \dots < a_m \le b.$$

A number of works ([1–5]) are devoted to estimating the length of the interval $[a, b]$ on which problem (1)–(2) has a unique solution. The following are known: the Vallée-Poussin estimate ([1])

$$\sum_{l=1}^{n} L_k \frac{(b-a)^k}{k!} \le 1; \tag{3}$$

the estimate of A. Yu. Levin ([2])

$$\sum_{k=1}^{n} \frac{L_k (b-a)^k}{2^k k \left[\frac{k-1}{2}\right]! \left[\frac{k}{2}\right]!} \le 1, \tag{4}$$

and, finally, Nehari' s integral estimate ([3])

$$\sum_{k=0}^{n-1}\left(\frac{b-a}{2}\right)^k\int_a^b |p_{k+1}(t)|\,dt \le 2. \tag{5}$$

Let us note that, for a fixed value of $(b-a)$, conditions (3)–(5) express the requirement of smallness of the coefficients L_i, or respectively

$$\int_a^b |p_i(t)|\,dt, \qquad i = 1, 2, \dots, n.$$

In the present paper each of the estimates (3)–(5) is sharpened. Conditions are obtained under which the coefficient $p_1(t)$ may assume arbitrarily large values on the interval $[a, b]$ owing to the smallness of the remaining coefficients, something not allowed by inequalities (3)–(5).

Theorem 1. *Problem (1)–(2), for arbitrary $A_{i,k}$, has a unique solution on the interval $[a, b]$ if*

$$\sum_{k=2}^{n} L_k \frac{(b-a)^k}{k!} \exp\{L_1(b-a)\} \le 1. \tag{6}$$

Estimate (6) sharpens estimate (3). Indeed, let us rewrite (3) and (6), respectively, in the form

$$\sum_{k=2}^{n} L_k \frac{(b-a)^k}{k!} \le 1 - L_1(b-a), \tag{7}$$

$$\sum_{k=2}^{n} L_k \frac{(b-a)^k}{k!} \leqslant \exp\{-L_1(b-a)\}. \tag{8}$$

The right-hand side of inequality (8), for any value of $(b-a)$, is greater than the right-hand side of inequality (7). Therefore inequality (6) determines a larger interval of solvability of problem (1)–(2) than inequality (3).

A consequence of Theorem 1 is the assertion that, for the equation

$$x^{(n)} + p_1(t)x^{(n-1)} = 0$$

problem (1)–(2) is solvable on any interval $[a, b]$, where the function $p_1(t)$ is continuous.

Theorem 2. *Problem (1)–(2), for arbitrary $A_{i,k}$, has a unique solution on the interval $[a, b]$, if*

$$\sum_{k=2}^{n} \frac{L_k(b-a)^k}{2^k k \left[\frac{k-1}{2}\right]! \left[\frac{k}{2}\right]!} \exp\left\{\frac{L_1}{2}(b-a)\right\} \leqslant 1. \tag{9}$$

Inequality (9) determines a larger interval of solvability of problem (1)–(2) than inequality (4).

Theorem 3. *Problem (1)–(2), for arbitrary A_{ik}, has a unique solution on the interval $[a,b]$, if*

$$\sum_{k=1}^{n-1} \frac{(b-a)^k \int_a^b |p_{k+1}(t)|\,dt}{2^k k \left[\frac{k-1}{2}\right]! \left[\frac{k}{2}\right]!} \exp\left\{\frac{1}{2}\int_a^b |p_1(t)|\,dt\right\} \leqslant 2. \tag{10}$$

Inequality (10) is a strengthening of inequality (5).

Corollary. *If a nontrivial solution of the equation*

$$x'' + p_1(t)x' + p_2(t)x = 0 \tag{11}$$

has two zeros on the interval $[a,b]$, then

$$(b-a)\int_a^b |p_2(t)|\,dt \exp\left\{\frac{1}{2}\int_a^b |p_1(t)|\,dt\right\} > 4. \tag{12}$$

If in equation (11) $p_1(t) = 0$, then inequality (12) passes into Lyapunov' s inequality

$$(b-a)\int_a^b |p_2(t)|\,dt > 4.$$

Therefore Theorem 3 may be regarded as a generalization of Lyapunov' s inequality for an equation of order n.

In conclusion we give the proof of Theorem 3. Suppose the contrary: let inequality (10) hold and, at the same time, let some nontrivial solution $x(t)$ of equation (1) have on the interval $[a,b]$ at least n zeros. Then on the interval $[a,b]$ there will be found a system of points

$$a \leqslant a_1 \leqslant a_2 \leqslant \cdots \leqslant a_{n-1} \leqslant c \leqslant b_{n-1} \leqslant \cdots \leqslant b_1 \leqslant b,$$

at which

$$x(a_1) = x'(a_2) = \cdots = x^{(n-2)}(a_{n-1}) = x^{(n-1)}(c) = \cdots = x(b_1) = 0.$$

Let

$$\sup_{a\le t\le c}|x^{(n-1)}(t)| = |x^{(n-1)}(\alpha)| = \mu \qquad (a \leqslant \alpha \leqslant c).$$

Then on the interval $[a, c]$ the estimate (2) holds:

$$\left|x^{(n-k-1)}(t)\right| \leqslant \mu \frac{(c-a)^k}{k\left[\frac{k-1}{2}\right]!\left[\frac{k}{2}\right]!}, \qquad k = 1, 2, \dots, n-1. \tag{13}$$

We rewrite equation (1) in the form

$$\left\{x^{(n-1)}(t)\exp\left[\int^t p_1(\xi)\,d\xi\right]\right\}' = -\sum_{k=1}^{n-1} p_{k+1}(t)x^{(n-k-1)}(t)\exp\left[\int^t p_1(\xi)\,d\xi\right].$$

Integrating this equation from a to c and applying estimate (13), we obtain the inequality

$$1 < \sum_{k=1}^{n-1}\frac{(c-a)^k\displaystyle\int_a^c |p_{k+1}(t)|\,dt}{k\left[\frac{k-1}{2}\right]!\left[\frac{k}{2}\right]!}\exp\int_a^c |p_1(\xi)|\,d\xi. \tag{14}$$

Similarly, for the interval $[c, b]$ we have

$$1 < \sum_{k=1}^{n-1}\frac{(b-c)^k\displaystyle\int_c^b |p_{k+1}(t)|\,dt}{k\left[\frac{k-1}{2}\right]!\left[\frac{k}{2}\right]!}\exp\int_c^b |p_1(\xi)|\,d\xi. \tag{15}$$

Denote the k-th term of the sums appearing on the right-hand sides of inequalities (14) and (15), respectively, by A_k^{1+k}, B_k^{1+k}, and introduce the quantity

$$\chi = \frac{1}{2}\int_a^b |p_1(t)|\,dt - \int_a^c |p_1(t)|\,dt.$$

Then

$$\frac{\displaystyle\int_a^b |p_{k+1}(t)|\,dt}{k\left[\frac{k-1}{2}\right]!\left[\frac{k}{2}\right]!}\exp\left\{\frac{1}{2}\int_a^b |p_1(t)|\,dt\right\} = \frac{\left[A_k\exp\frac{\chi}{1+k}\right]^{1+k}}{(c-a)^k} +$$

$$+\frac{\left[B_k \exp\left(-\frac{\chi}{1+k}\right)\right]^{1+k}}{(b-c)^k} \geq \frac{\left[A_k \exp\frac{\chi}{1+k} + B_k \exp\left(-\frac{\chi}{1+k}\right)\right]^{1+k}}{(b-a)^k}$$

or

$$\sum_{k=1}^{n-1} \frac{(b-a)^k \displaystyle\int_a^b |p_{k+1}(t)|\,dt}{2^k k \left[\frac{k-1}{2}\right]! \left[\frac{k}{2}\right]!} \exp\left\{\frac{1}{2}\int_a^b |p_1(t)|\,dt\right\} \geq$$

$$\geq \sum_{k=1}^{n-1} 2^{-k}\left[A_k \exp\frac{\chi}{1+k} + B_k \exp\left(-\frac{\chi}{1+k}\right)\right]^{1+k} > 2 \qquad (16)$$

under conditions (14) and (15), i.e., under the condition that

$$\sum_{k=1}^{n-1} A_k^{1+k} > 1, \qquad \sum_{k=1}^{n-1} B_k^{1+k} > 1.$$

Inequality (16) contradicts inequality (10). The theorem is proved.

Bauman Higher Technical School

Received
10 XII 1966

REFERENCES

1. Ch. J. de la Vallée-Poussin, *J. math. pures et appl.*, **9**, No. 8, 125 (1929).

2. A. Yu. Levin, *Matem. sborn.*, **64**, No. 3, 396 (1964).

3. Z. Nehari, *Studies in Mathematical Analysis and Related Topics*, 1962, p. 256.

4. A. Yu. Levin, *DAN*, **153**, No. 6, 1257 (1963).

5. G. A. Bessmertnykh, A. Yu. Levin, *DAN*, **144**, No. 3, 471 (1962).

Note: Figure translations are in progress. See original paper for figures.