THE SHIMURA CONJECTURE

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Abstract

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MATHEMATICS

A. N. ANDRIANOV

THE SHIMURA CONJECTURE

FOR THE SIEGEL MODULAR GROUP OF GENUS 3

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1. Let Z be the ring of rational integers;

$$J_n = \begin{pmatrix} 0 & 1_n \\ -1_n & 0 \end{pmatrix},$$

where 1_n is the identity matrix of order n; $\Gamma = \mathrm{sp}(n, Z)$ is the Siegel modular group of genus n. Denote by S the set of all matrices B of order $2n$ with elements from Z, for which ${}^tBJ_{nB} = r(B)J_n$, where $r(B) \in Z$, $r(B) > 0$. Let L be the free Z-module generated by all double classes $\Gamma B \Gamma$ for $B \in S$. Introduce multiplication in L ([1]). Let L_p, where p is a prime number, be the subring of L generated by those $\Gamma B \Gamma$ for which $r(B)$ is a power of p. Then L_p is a polynomial ring over Z in $n+1$ elements

$$T_{0,n} = T(\underbrace{1, \dots, 1}_{n}, \underbrace{p, \dots, p}_{n}),$$

$$T_{i,n} = T(\underbrace{1, \dots, 1}_{n-i}, \underbrace{p, \dots, p}_{i}, \underbrace{p^2, \dots, p^2}_{n-i}, \underbrace{p, \dots, p}_{i}) \quad (1 \le i \le n),$$

where $T(p^{\alpha_1}, \dots, p^{\alpha_{2n}})$ denotes the double class containing the diagonal matrix with the diagonal indicated in parentheses; these elements are algebraically independent ([1,3]).

Introduce the local Hecke series of the group Γ, putting

$$D_p(s) = \sum_{(\Gamma B \Gamma) \in L_p} (\Gamma B \Gamma)\, r(B)^{-s}.$$

Shimura ([1]) conjectured that $D_p(s)$ is a rational function of $X = p^{-s}$; more precisely, he assumed that $D_p(s) = E_n(X) \cdot F_n(X)^{-1}$, where $E_n(X)$ and $F_n(X)$ are polynomials in X of degrees $2^n - 2$ and 2^n, respectively. For $n = 1$ the function $D_p(s)$ was computed by Hecke ([2]); for $n = 2$ by Shimura ([1]).

The aim of the present note is to compute the function $D_p(s)$ for $n = 3$. We show that in this case

$$D_p(s) = \left[\sum_{n=0}^{6}(-1)^{n+1}e(n)X^n\right] \times \left[\sum_{n=0}^{8}(-1)^n f(n)X^n\right]^{-1}, \tag{1}$$

where

$$e(0) = 1, \quad e(1) = 0, \quad e(2) = p^2(T_{2,3} + (p^4 + p^2 + 1)T_{3,3}),$$

$$e(3) = p^4(1+p)T_{0,3}T_{3,3}, \quad e(4) = p^7(T_{2,3}T_{3,3} + (p^4 + p^2 + 1)T_{3,3}^2),$$

$$e(5) = 0, \quad e(6) = -p^{15}T_{3,3}^3; \quad f(0) = 1, \quad f(1) = T_{0,3}, \quad f(2) = pT_{1,3}+$$

$$+p(p^2+1)T_{2,3} + (p^5 + p^4 + p^3 + p)T_{3,3}, \quad f(3) = p^3(T_{0,3}T_{2,3} + T_{0,3}T_{3,3}),$$

$$f(4) = p^6T_{0,3}^2T_{3,3} + p^6T_{2,3}^2 - 2p^7T_{1,3}T_{3,3} - 2p^6(p-1)T_{2,3}T_{3,3} - (p^{12} + 2p^{11}$$

$$+2p^{10} + 2p^7 - p^6)T_{3,3}^2, \quad f(5) = p^6T_{3,3}f(3), \quad f(6) = p^{12}T_{3,3}^2 f(2),$$

$$f(7) = p^{18}T_{3,3}^3 f(1), \quad f(8) = p^{24}T_{3,3}^4.$$

We outline the proof of formula (1).

2. Let Q_p be the field of p-adic numbers; Z_p the ring of integral p-adic numbers; G_p the group of all matrices $A \in GL(2n, Q_p)$ for which ${}^tAJ_{nA} = r(A)J_n$, where $r(A) \in Q_p$; and Γ_p the subgroup of G_p consisting of matrices A with coefficients in Z_p, for which $r(A)$ is a p-adic unit. Let $L(G_p, \Gamma_p)$ be the algebra over the field of complex numbers $\mathbf{C}$, formed by all complex-valued continuous functions φ on G_p with compact support,

which are constant on the double classes of G_p with respect to Γ_p, where multiplication is defined as convolution:

$$\varphi * \psi(A) = \int_{G_p} \varphi(AB^{-1})\psi(B)\,dB;$$

here dB is an invariant Haar measure on G_p, normalized by the condition

$$\int_{\Gamma_p} dB = 1.$$

Let $B \in S$ (see paragraph 1) and let $r(B)$ be a power of the number p. For each such B denote by $\overline{\Gamma_{pB}\Gamma_p}$ the function on G_p equal to 1 on the set $\Gamma_{pB}\Gamma_p$ and to 0 outside this set. From the theory of elementary divisors for the symplectic group ([3]) it follows that the mapping

$$B \to \overline{\Gamma_{pB}\Gamma_p},$$

extended by linearity to L_p, defines an isomorphism of the ring L_p onto a subring $\overline{L}_p$ of the ring $L(G_p, \Gamma_p)$. Recall that a function ω on G_p is called zonal spherical if it is constant on the double classes of G_p with respect to Γ_p and the mapping

$$\varphi \to \int_{G_p} \varphi(A)\omega(A^{-1})\,dA = \hat{\omega}(\varphi)$$

is a nontrivial homomorphism of the algebra $L(G_p, \Gamma_p)$ into $\mathbf{C}$ ([3,4]).

Using the fact that $\overline{L}_p \simeq L_p$ is a polynomial ring over Z, it is easy to see that, in order to compute the function $D_p(s)$, it suffices to compute its "Fourier transform," i.e., the function

$$\hat{D}_p(s) = \sum_{\overline{\Gamma_{pB}\Gamma_p} \in \overline{L}_p} \hat{\omega}(\overline{\Gamma_{pB}\Gamma_p}) r(B)^{-s} =$$

$$= \sum_{\overline{\Gamma_{pB}\Gamma_p} \in \overline{L}_p} \left(\int_{G_p} \overline{\Gamma_{pB}\Gamma_p}(A)\omega(A^{-1})\,dA \right) r(B)^{-s} =$$

$$= \int_{G_p} \left(\sum r(B)^{-s} \overline{\Gamma_{pB}\Gamma_p} \right)(A)\omega(A^{-1})\,dA = \int_{S_p} \omega(A^{-1}) r_0(A)^{-s}\,dA = \xi(s, \omega),$$

where ω is an arbitrary zonal spherical function on G_p, $r_0(A) = p^{\nu_p(r(A))}$, and S_p is the set of all $A \in G_p$ with coefficients in Z_p. The function $\xi(s, \omega)$ is called the ζ-function of the group G_p ([3,4]).

3. We describe the set of zonal spherical functions on G_p (3). Let H_p be the subgroup of all matrices in G_p of the form

$$C = \begin{pmatrix} g & e \\ 0 & {}^t g^{-1} p^k \end{pmatrix} \begin{matrix} \}n \\ \}n \end{matrix}, \tag{2}$$

where g is a triangular matrix whose entries below the main diagonal are zeros and whose entries on the main diagonal are powers of p. Define a function ψ on H_p by putting

$$\psi(C) = \lambda_1^{a_1} \cdots \lambda_n^{a_n} \beta^k,$$

where $\lambda_1, \dots, \lambda_n, \beta$ are a fixed set of nonzero complex numbers and $a_1, \dots, a_n$ are determined from the condition that the diagonal of g is equal to $(p^{a_1}, \dots, p^{a_n})$. Since $\psi(H_p \cap \Gamma_p) = 1$ and $G_p = \Gamma_{pH} p = H_p \Gamma_p$ (3), we can extend ψ to G_p, putting, for $A = UC$, $U \in \Gamma_p$, $C \in H_p$, $\psi(A) = \psi(C)$. Then the function

$$\omega_\psi(A) = \int_{\Gamma_p} \psi(AU)\, dU$$

is a zonal spherical function on G_p, and every zonal spherical function on G_p can be obtained in the indicated way by a suitable choice of the parameters $\lambda_1, \dots, \lambda_n, \beta$ (3).

4. Let $\omega = \omega_\psi$ be a zonal spherical function on G_p with parameters $\lambda_1, \dots, \lambda_n, \beta$. Then it is easy to see that

$$\xi(s, \omega) = \int_{S_p} r_0^{-s}(A) \left(\int_{\Gamma_p} \psi(A^{-1}U)\, dU \right) dA = \int_{S_p} r_0(A)^{-s} \psi(A^{-1})\, dA.$$

Let $S_{p,k}$, for $k = 0, 1, 2, \dots$, denote the set of all $A \in S_p$ for which $r_0(A) = p^k$, and let $S_{p,k} = \bigcup \alpha_{ik} \Gamma_p$ be a representation of $S_{p,k}$ as a disjoint union of right classes modulo Γ_p. Put $p^{-s} = X$; then

$$\xi(s, \omega) = \sum_{k=0}^{\infty} X^k \int_{S_{p,k}} \psi(A^{-1})\, dA = \sum_{k=0}^{\infty} \left(\sum_i \psi(\alpha_{ik}^{-1}) \right) X^k. \tag{3}$$

As is known (5), as the α_{ik} one may take a set of matrices of the form (2), where g runs through all matrices of the form

$$g = \begin{pmatrix} p^{\alpha_1} & c_{12} & \cdots & c_{1n} \\ 0 & p^{\alpha_2} & & c_{2n} \\ \cdot & & \cdot & \\ \cdot & & & \cdot \\ 0 & 0 & & p^{\alpha_n} \end{pmatrix} \tag{4}$$

with $\alpha_i \geq 0$, $c_{i,i+1}, \dots, c_{i,n}$ belonging to the reduced system of residues modulo p^{α_i} for $i = 1, \dots, n-1$ (such matrices will be called reduced), and such that the matrix $p^k g^{-1}$ has integral coefficients, and where, for fixed g, e runs through a complete set of representatives of the classes into which the set of integral matrices of order n, e, satisfying the condition $g^t e = e^t g$, is divided with respect to the equivalence relation $e \sim e_1 \leftrightarrow e - e_1 = g^t t$ with integral matrix t. Let g be an integral matrix of order n, $\det g = p^\nu$, and let $p^{\nu_i(g)}$, for $i = 1, \dots, n$, be the greatest common divisor of the minors of order i of the matrix g, $\nu_0(g) = 0$. Put, for $i = 1, \dots, n$, $\beta_i(g) = \nu_i(g) - \nu_{i-1}(g)$; the numbers $\beta_i(g)$ are called the elementary divisors of g. One can verify that the number of classes into which the matrices e are divided with respect to g is equal to $p^{n\beta_1(g)+\dots+\beta_n(g)}$. We also note that the integrality of $p^k g^{-1}$ is equivalent to the condition $\beta_n(g) \leq k$. Thus,

$$\sum_i \psi(\alpha_{ik}^{-1}) = \left(\sum_{0 \leq \beta_1 \leq \dots \leq \beta_n \leq k} p^{n\beta_1 + (n-1)\beta_2 + \dots + \beta_n} \gamma(\lambda_1, \dots, \lambda_n; \beta_1, \dots, \beta_n) \right) \beta^{-k},$$

where

$$\gamma(\lambda_1, \dots, \lambda_n; \beta_1, \dots, \beta_n) = \sum_g \lambda_1^{-\alpha_1} \cdots \lambda_n^{-\alpha_n},$$

and here the summation extends over all reduced g of the form (4) with elementary divisors $\beta_1, \dots, \beta_n$.

5. Let now $F = GL(n, \mathbf{Q}_p)$, $U = GL(n, \mathbf{Z}_p)$. Analogously to item 2, one can define the algebra $L(F, U)$ and the zonal spherical functions $\varepsilon = \varepsilon_\psi$ with parameters $\lambda_1, \dots, \lambda_n$ on F [4]. Let $\chi(\beta_1, \dots, \beta_n)$, where the β_i are integers, $\beta_1 \leq \dots \leq \beta_n$, be the characteristic function of the double class of F modulo U containing the diagonal matrix with diagonal $(p^{\beta_1}, \dots, p^{\beta_n})$. Obviously, $\chi(\beta_1, \dots, \beta_n) \in L(F, U)$. Let ε be a zonal spherical function on F with parameters $\lambda_1, \dots, \lambda_n$, and let $\varphi \mapsto \breve{\varepsilon}(\varphi)$ be the homomorphism $L(F, U)$ into $\mathbf{C}$ that it determines. Then, using the results of [4], it is not difficult to show that

$$\breve{\varepsilon}(\chi(\beta_1, \dots, \beta_n)) = \gamma(\lambda_1, \dots, \lambda_n; \beta_1, \dots, \beta_n).$$

Thus, taking into account the results of Sec. 4, we see that in order to compute the series (3) it is sufficient to compute the series

$$Z_n(X) = \sum_{k=0}^{\infty} \left(\sum_{0 \leqslant \beta_1 \leqslant \dots \leqslant \beta_n \leqslant k} \chi(\beta_1, \dots, \beta_n) p^{n\beta_1 + (n-1)\beta_2 + \dots + \beta_n} \right) X^k. \tag{5}$$

6. The ring $L(F,U)$ is the ring of polynomials over C in the functions $\pi_{1,n} = \chi(0,\dots,0,1),\dots,\pi_{n,n} = \chi(1,\dots,1)$ and $\chi(-1,\dots,-1)$, the first n of which are algebraically independent over C [(4)]. However, in order to compute the series (5) we need to know the multiplication table for the functions χ. In what follows we restrict ourselves to the case $n = 3$. It can be shown that in this case the following multiplication table holds:

$$\pi_{3,3}\chi(\beta_1,\beta_2,\beta_3) = \chi(\beta_1+1,\beta_2+1,\beta_3+1),$$

$$\pi_{1,3}\chi(0,e,n) = \chi(0,e,n+1) + \alpha(n-e)\chi(0,e+1,n) +$$

$$+(p\alpha(e) + \beta(e,n))\chi(1,e,n),$$

$$\pi_{2,3}\chi(0,e,n) = \chi(0,e+1,n+1) + \alpha(e)\chi(1,e,n+1) +$$

$$+(p\alpha(n-e) + \beta(e+1,n))\chi(1,e+1,n),$$

where $\alpha(0) = 0$, $\alpha(1) = p+1$, $\alpha(m) = p$ if $m > 1$, and $\beta(k,m) = 1$ or $\beta(k,m) = 0$ according as $(k,m) = (1,1)$ or $(k,m) \neq (1,1)$.

7. In the notation introduced above, the formula holds

$$\begin{aligned} Z_3(X) = {} & [1 - C(2)X^2 + C(3)X^3 - C(4)X^4 + p^{15}\chi(3,3,3)X^6]\times \\ & \times [(1-X)(1 - p\pi_{1,3}X + p^3\pi_{2,3}X^2 - p^6\pi_{3,3}X^3)\times \\ & \times (1 - p^3\pi_{2,3}X + p^7\pi_{1,3}\pi_{3,3}X^2 - p^{12}\pi_{3,3}^2X^3)(1 - p^6\pi_{3,3}X)]^{-1}, \end{aligned} \tag{6}$$

where

$$C(2) = p^6\pi_{3,3}\pi_{1,3} + p^4(1+p+p^2)\pi_{3,3} + p^2\pi_{2,3}, \qquad C(4) = p^5\pi_{3,3}C(2),$$

$$C(3) = p^4(p+1)(\pi_{3,3} + p\pi_{3,3}\pi_{1,3} + p^3\pi_{3,3}\pi_{2,3} + p^6\pi_{3,3}^2).$$

Concerning the proof of this formula, we note that all recurrence relations arising in this connection are proved by induction using the rules for multiplying the functions χ given in the preceding section.

8. Let us now return to the function $D_p(s)$. Let ε be the same as in Sec. 5. Applying to the left- and right-hand sides of relation (6) the homomorphism $\hat{\varepsilon}$ of the algebra $L(F,U)$ and replacing X by $X\beta^{-1}$, we obtain an expression for $\hat{D}_p(s) = \xi(s,\omega)$ in the form of a rational fraction, explicit formulas for the coefficients of whose numerator and denominator are obtained from the easily proved relations

$$\hat{\varepsilon}(\pi_{1,3}) = \lambda_1^{-1} + p\lambda_2^{-1} + p^2\lambda_3^{-1} = \varphi_1,$$

$$\hat{\varepsilon}(\pi_{2,3}) = (\lambda_1\lambda_2)^{-1} + p(\lambda_1\lambda_3)^{-1} + p^2(\lambda_2\lambda_3)^{-1} = \varphi_2,$$

$$\hat{\varepsilon}(\pi_{3,3}) = (\lambda_1\lambda_2\lambda_3)^{-1} = \varphi_3.$$

Formula (1) is now established by eliminating the parameters $\lambda_1, \lambda_2, \lambda_3, \beta$ from the resulting expression for $\hat{D}_p(s)$ and the directly verifiable identities

$$\hat{\omega}(T_{0,3}) = \beta^{-1}[1 + p\varphi_1 + p^3\varphi_2 + p^6\varphi_3],$$

$$\hat{\omega}(T_{1,3}) = \beta^2[\varphi_1 + (p^2-1)\varphi_2 - (p^4+2p^3)\varphi_3 + p^4(p^2-1)\varphi_1\varphi_3 + p^8\varphi_2\varphi_3 + p^3\varphi_1\varphi_2],$$

$$\hat{\omega}(T_{2,3}) = \beta^{-2}[\varphi_2 + (p^3-1)\varphi_3 + p^4\varphi_3\varphi_1], \qquad \hat{\omega}(T_{3,3}) = \beta^{-2}\varphi_3,$$

where $\hat{\omega}$ is the homomorphism of the algebra $L(G_p, \Gamma_p)$ into C, defined with the aid of the function ω (see Sec. 2).

Leningrad Branch
of the V. A. Steklov Mathematical Institute
Academy of Sciences of the USSR

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REFERENCES

1. G. Shimura, Proc. Nat. Acad. Sci. U.S.A., **49**, 824 (1963).

2. E. Hecke, Math. Ann., **114**, 1 (1937).

3. I. Satake, Inst. Hautes Étude Sci. Publ. Math., No. 18 (1963).

4. T. Tamagawa, Ann. Math., **77**, 387 (1963).

5. H. Maass, Math. Ann., **124**, 1, 87 (1951).

Note: Figure translations are in progress. See original paper for figures.