ON PERIODIC SOLUTIONS OF ONE CLASS OF DIFFERENTIAL EQUATIONS WITH HYSTERESIS NONLINEARITY

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Abstract

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MATHEMATICS

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ON PERIODIC SOLUTIONS OF ONE CLASS OF DIFFERENTIAL EQUATIONS WITH HYSTERESIS NONLINEARITY

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1. In a Banach space E consider an operator $\mathcal{L}$ with domain of definition $D(\mathcal{L})$. Denote by E_0 the closure of the linear manifold $D(\mathcal{L})$. We shall assume (see (1)) that the operator $\mathcal{L}$ is the infinitesimal operator of a semigroup, bounded in E, of operators $T(t)$, which is strongly continuous for $t > 0$ and $\|T(t)u - u\| \to 0$ $(u \in E_0)$ as $t \to +0$. Further, we shall assume that $\|T(t)\| \leq a < \infty$ $(0 < t \leq 1)$ and that the values of the operators $T(t)$ $(t > 0)$ belong to E_0. Finally, we shall suppose that the operator $\mathcal{L}$ has such a simple negative eigenvalue λ_0 that for the remaining points λ of the spectrum the inequality $\operatorname{Re}\lambda < \lambda_0$ holds.

Fix any two numbers d_1 and d_2, elements $f_i, \psi_i \in E$ $(i = 1, 2)$, and a continuous linear functional $l(u)$. It is assumed below that $\psi_1 - \psi_2 \in E_0$ and that the numbers d_i and $l(v_i)$ $(i = 1, 2;\ v_i = -\mathcal{L}^{-1} f_i + \psi_i)$ are related by the inequalities

$$l(v_1) > d_1 > d_2 > l(v_2). \tag{1}$$

Definition 1. The function

$$u(t) = T(t)u_0 + v_i - T(t)v_i \quad (i = 1, 2)$$

will be called a **solution of the equation**

$$du/dt - \mathcal{L}u = f_i, \tag{2}$$

satisfying the initial condition $u(0) = u_0$.

We now consider the equation

$$du/dt - \mathcal{L}u = \begin{cases} f_1, \\ f_2. \end{cases} \tag{3}$$

Definition 2. We shall say that a function $u(t)$, continuous for $0 \leq t < \infty$, is a **solution of equation** (3) if for each $T > 0$ there exists only a finite number of values $t:\ t_0, t_1, \dots, t_m$ $(t_0 = 0,\ t_i < t_{i+1} \leq T,\ i = 0, \dots, m-1)$ such that: 1) either $l[u(t_{i+1})] = d_1$, and then for $t_i < t < t_{i+1}$ the function $u(t)$ is a solution of equation (2) with right-hand side f_1, while $l[u(t)] < d_1$ for $t_i \leq t < t_{i+1}$; 2) or $l[u(t_{i+1})] = d_2$, and then for $t_i < t < t_{i+1}$ the function $u(t)$ is a solution of equation (2) with right-hand side f_2, while $l[u(t)] > d_2$ for $t_i \leq t < t_{i+1}$.

We shall be interested in conditions under which equation (3) has periodic solutions. An analogous problem for equations of a special form was studied in applied papers ($^{2-5}$) and by exact methods in (^{6}).

2. Definition 3. A periodic solution $u_0(t)$ of equation (3) will be called a **periodic solution with $2n$ switchings**,

if one can specify a set of $2n$ positive numbers $\tau_1, \dots, \tau_{2n}$ such that on each of the intervals

$$(\tau_0 + \tau_1 + \dots + \tau_i,\ \tau_0 + \tau_1 + \dots + \tau_{i+1}) \qquad (\tau_0 = 0;\ i = 0, \dots, 2n-1)$$

the function $u_0(t)$ is a solution of equation (2) with right-hand side f_1, if i is even, and a solution of equation (2) with right-hand side f_2, if i is odd, and moreover

$$u_0(0) = u_0(\tau_1 + \dots + \tau_{2n}),$$

while

$$u_0(\tau_1 + \dots + \tau_i) \neq u_0(0) \quad \text{for } i = 1, 2, \dots, 2n-1.$$

Put

$$\alpha(t) = d_1 - d_2 - l[(I - T(t))(v_1 - v_2)] \qquad (0 \leq t < \infty).$$

It follows from our assumptions that $\alpha(0) > 0$, and

$$\alpha(t) \to d_1 - d_2 - l(v_1 - v_2) < 0$$

as $t \to \infty$. Therefore the equation $\alpha(t) = 0$ has at least one root $t = t^*$. We shall assume that this root is unique, and shall call this assumption **condition (A)**.

Theorem 1. *Let condition (A) be satisfied. Let*

$$T(kt_0)(v_1 - v_2) \in D(\mathcal{L}^k) \qquad (k = 1, 2).$$

Finally, suppose that for $t > t_0$ the inequalities

$$(-1)^k l[\mathcal{L}^{kT}(kt)(I - T(t_0))^{2-k}(v_1 - v_2)] > 0 \qquad (k = 1, 2),$$

hold, where t_0 is any number satisfying the inequalities $0 < t_0 \leq t^*$.

Then equation (3) has at least one periodic solution with two switchings, and

$$\min(\tau_1, \tau_2) > t^*. \tag{4}$$

3. **Definition 4.** We shall say that a periodic solution $u_0(t)$ of equation (3) belongs to the class A_{t_0} if

$$\max \tau_i \geq t_0 \qquad (1 \leq i \leq 2n).$$

Theorem 2. *Let condition (A) be satisfied. Let*

$$T(kt_0)(v_1 - v_2) \in D(\mathcal{L}^k) \qquad (k = 1, 2, 3).$$

Finally, suppose that for $t > t_0$ the inequalities

$$(-1)^k l[\mathcal{L}^{kT}(kt)(I - T(t_0))^{3-k}(v_1 - v_2)] > 0 \qquad (k = 1, 2, 3)$$

hold.

Then equation (3) has, in the class A_{t_0}, a unique periodic solution. This periodic solution has two switchings; for it inequality (4) is satisfied.

4. **Definition 5.** We shall say that a periodic solution $u_0(t)$ $(l[u_0(0)] = d_2)$ is stable if there exists a $\delta > 0$ such that

$$\|u_0(t) - u(t + c)\| < \omega(t),$$

where c is some constant depending only on the solution $u(t)$, while $\omega(t) \to 0$ as $t \to \infty$, for all solutions $u(t)$ of equation (3) whose initial conditions satisfy the inequality

$$\|u_0(0) - u(0)\| < \delta$$

and the equality

$$l[u(0)] = d_2.$$

From our assumptions concerning the operator $\mathcal{L}$ it follows that the operator $T(t)$ has an invariant subspace E_1 such that each element $u \in E$ is uniquely representable in the form

$$u = l_0(u)v_0 + u_1,$$

where $u_1 \in E_1$, v_0 is an eigenvector corresponding to the eigenvalue λ_0, and $l_0(u)$ is a linear functional.

Theorem 3. *Let*

$$T(t_0)u \in D(\mathcal{L})$$

for all $u \in E$. Let

$$l_0(v_1 - v_2) \neq 0.$$

Finally, let the operator $T(t_0)$ be completely continuous.

Then, under the conditions of Theorem 2, the unique periodic solution of class A_{t_0} is stable.

5. In the space $C(\Omega)$ (of functions continuous in the closed domain $\overline{\Omega}$ with boundary S) define the operator

$$\mathcal{L}u = \sum_{i,k=1}^{n} a_{ik}(x)\frac{\partial^2 u}{\partial x_i \partial x_k} + \sum_{i=1}^{n} a_i(x)\frac{\partial u}{\partial x_i} + a(x)u. \tag{5}$$

on sufficiently smooth functions satisfying the homogeneous boundary condition

$$\Gamma u = 0 \quad (x \in S), \tag{6}$$

where either $\Gamma u \equiv u(x)$, or $\Gamma u = (b(x), \partial u/\partial x) + \sigma(x)u$, and the continuous vector field $b(x)$ at each point x of the boundary S forms an acute angle with the direction of the exterior normal. If $\overline{\Omega} = [a, b]$, then we shall assume that

$$\Gamma u = \begin{cases} \alpha_1 u'_x(a) + \beta_1 u(a), \\ \alpha_2 u'_x(b) + \beta_2 u(b), \end{cases}$$

where $\alpha_1 \leq 0,\ \alpha_2 \geq 0$ and $\alpha_i^2 + \beta_i^2 > 0$ $(i = 1, 2)$.

Under the usual smoothness assumptions on the coefficients of the differential expressions (5) and (6), with sufficiently smooth boundary S of the domain Ω, and under the conditions of uniform ellipticity (see (7)), the operator (5)–(6) generates in the space $C(\Omega)$ a strongly continuous, for $t > 0$, semigroup of bounded operators $T(t)$, and

$$\lim_{t \to +0} T(t)u(x) = u(x)$$

for $u(x) \in C_0(\Omega)$ (by $C_0(\Omega)$ is denoted the closure of the linear manifold of functions satisfying the boundary condition (6)).

We also note that the operator (5)–(6) has a simple real eigenvalue λ_0 such that for the remaining eigenvalues λ the inequality $\operatorname{Re}\lambda < \lambda_0$ holds.

Let $l(u)$ be some continuous linear functional in $C(\Omega)$, and let d_1 and d_2 be two numbers.

Consider the partial differential equation

$$\partial u/\partial t - \mathcal{L}u = \begin{cases} f_1(x), \\ f_2(x) \end{cases} \qquad (x \in \Omega) \tag{7}$$

with boundary conditions

$$\Gamma u = \begin{cases} \varphi_1(x), \\ \varphi_2(x) \end{cases} \quad (x \in S). \tag{8}$$

Using the functional $l(u)$ and the numbers d_1, d_2, we define the solutions of problem (7)–(8) analogously to how this was done in Definition 2.

Denote by $\psi_i(x)$ $(i = 1, 2)$ the solutions of the boundary-value problems

$$\mathscr{L}u = 0 \quad (x \in \Omega), \qquad \Gamma u = \varphi_i(x) \quad (x \in S).$$

Then, by virtue of the properties of the operator (5)–(6) described above, equation (7), together with the boundary conditions (8), can be regarded as an ordinary differential equation of the form (3) in the Banach space $C(\Omega)$. Consequently, Theorems 1-3 can be applied to problem (7)–(8). In studying the spectrum of the corresponding operators and in verifying condition (A), the following lemmas are useful.

Lemma 1. *In order that the spectrum of the operator (5)–(6) lie in the left half-plane, it is necessary and sufficient that there exist a nonnegative function* $v(x)$ *such that*

$$\mathscr{L}v(x) \leq 0 \quad (x \in \Omega), \tag{9}$$

$$\Gamma v(x) \geq 0 \quad (x \in S), \tag{10}$$

and either (9) or (10) does not reduce to an identity.

Lemma 2. *Let the spectrum of the operator (5)–(6) lie in the left half-plane. Let inequalities (1) be satisfied with some nonnegative functional* $l(u)$. *Finally, let*

$$f_1(x) \geq f_2(x) \quad (x \in \Omega), \qquad \varphi_1(x) \geq \varphi_2(x) \quad (x \in S).$$

Then condition (A) is satisfied.

In conclusion, we give several consequences of the general Theorems 1-3.

Theorem 4. *Suppose that, under the conditions of Lemma 2,* $\varphi_1(x) \equiv \varphi_2(x)$ $(x \in S)$. *Suppose*

$$\mathscr{L}(f_1 - f_2) \leqslant 0 \quad (x \in \Omega)$$

and

$$\Gamma(f_1 - f_2) \geqslant 0 \quad (x \in S).$$

Then problem (7)–(8) has at least one periodic solution.

We note that Theorem 4 contains the corresponding result of [6].

Theorem 5. *Suppose that, under the conditions of Theorem 4,*

$$\Gamma\mathcal{L}(f_1 - f_2) = 0 \quad (x \in S).$$

Suppose

$$\mathcal{L}^2(f_1 - f_2) \geqslant 0 \quad (x \in \Omega).$$

Then problem (7)–(8) has a unique periodic solution; this periodic solution is stable.

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Note: Figure translations are in progress. See original paper for figures.