DUALITY THEORY FOR CONCAVE-CONVEX GAMES

MATHEMATICS

1967

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Abstract

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UDC 518.9

MATHEMATICS

V. N. LEBEDEV, N. T. TYNYANSKII

DUALITY THEORY FOR CONCAVE-CONVEX GAMES

(Presented by Academician Yu. V. Linnik, 29 VIII 1966)

1. Conjugate sets. On $L = C \times D$ a function $f(x, y)$ is defined, concave in $x \in C \subset E^m$ and convex in $y \in D \subset E^n$. The sets C and D are convex. The function $f(x, y)$ is assumed to be closed on L, i.e., for any point $(x^0, y^0) \in L$ at which the limits $\lim\limits_{y \to y^0} f(x^0, y)$ and $\lim\limits_{x \to x^0} f(x, y^0)$ are finite, the value $f(x, y)$ is defined and equals

$$f(x^0, y^0) = \lim_{y \to y^0} f(x^0, y) = \lim_{x \to x^0} f(x, y^0).$$

Consider the sets

$$[f^-, L] = \{\{a, x, y\} \in E^{m+n+1} \mid x \in C,\ y \in D,\ a \le f(x, y)\},$$

$$[f^+, L] = \{\{b, x, y\} \in E^{m+n+1} \mid x \in C,\ y \in D,\ b \ge f(x, y)\}.$$

For fixed $y \in D$ the set

$$[f_y^-, C] = \{\{a, x\} \in E^{m+1} \mid x \in C,\ a \le f(x, y)\}$$

is closed and convex. Similarly, for fixed $x \in C$ the set

$$(f_x^+, D) = \{\{b, y\} \in E^{n+1} \mid y \in D,\ b \ge f(x, y)\}$$

is also closed and convex.

For the set $[f_y^-, C]$ define the conjugate set

$$[f_y^-, C]^* = \{\{\alpha, \gamma\} \in E^{m+1} \mid (a, x) \in [f_y^-, C],\ a + \gamma x \le \alpha\}.$$

Put

$$\Gamma_y = \{\gamma \in E^m \mid \sup_{x\in C}[f(x,y)+\gamma x] < \infty\},$$

$$L_x^* = \bigcup_{y\in D} \Gamma_y = \{\{\gamma, y\} \in E^{m+n} \mid y \in D,\ \gamma \in \Gamma_y\}.$$

On the set L_x^* define the function

$$\varphi(\gamma, y) = \sup_{x\in C}[f(x,y)+\gamma x]. \tag{1}$$

Then for fixed $y \in D$

$$[f_y^-, C]^* = \{\{\alpha, \gamma\} \in E^{m+1} \mid \gamma \in \Gamma_y,\ \alpha \geq \varphi(\gamma, y)\}.$$

The set

$$[f^-, L]_x^* = [\varphi, L_x^*] = \bigcup_{y\in D} [f_y^-, C]^* =$$

$$= \{\{\alpha, \gamma, y\} \in E^{m+n+1} \mid (\gamma, y) \in L_x^*,\ \alpha \geq \varphi(\gamma, y)\} \tag{2}$$

will be called the set conjugate in x to the set $[f^-, L]$.

Similarly, for the set $[f_x^+, D]$ the conjugate set is

$$[f_x^+, D]^* = \{\{\beta, \delta\} \in E^{n+1} \mid (b, y) \in [f_x^+, D],\ b - \delta y \geq \beta\},$$

$$\Delta_x = \{\delta \in E^n \mid \inf_{y\in D}[f(x,y) - \delta y] > -\infty\},$$

$$L_y^* = \bigcup_{x\in C} \Delta_x = \{\{x, \delta\} \in E^{m+n} \mid x \in C,\ \delta \in \Delta_x\}, \tag{3}$$

$$\psi(x, \delta) = \inf_{y\in D}[f(x,y) - \delta y],$$

$$[f_x^*, D]^* = \{(\beta, \delta) \in E^{n+1} \mid \delta \in \Delta_x,\ \beta \leqslant \psi(x, \delta)\},$$

$$\begin{aligned}[f^+, L]^*_y = [\psi, L^*_y] &= \bigcup_{x\in C} [f^*_x, D]^* \\ &= \{(\beta, x, \delta) \in E^{m+n+1} \mid (x, \delta) \in L^*_y,\ \beta \leqslant \psi(x, \delta)\}. \end{aligned} \tag{4}$$

Next, denoting by Γ' the projection of the set L^*_x onto E^m and fixing $\gamma \in \Gamma'$, put

$$D_\gamma = \{y \in E^n \mid (\gamma, y) \in L^*_x\},$$

$$[\varphi_\gamma, D_\gamma] = \{(\alpha, y) \in E^{n+1} \mid y \in D_\gamma,\ \alpha \geqslant \varphi(\gamma, y)\}.$$

Similarly to the preceding,

$$\Delta_\gamma = \left\{\delta \in E^n \,\middle|\, \inf_{y\in D_\gamma} [\varphi(\gamma, y) - \delta y] > -\infty\right\},$$

$$\Lambda' = \bigcup_{\gamma\in\Gamma'} \Delta_\gamma = \{(\gamma, \delta) \in E^{m+n} \mid \gamma \in \Gamma',\ \delta \in \Delta_\gamma\}, \tag{5}$$

$$g'(\gamma, \delta) = \inf_{y\in D_\gamma} [\varphi(\gamma, y) - \delta y],$$

$$[\varphi_\gamma, D_\gamma]^* = \{(\zeta, \delta) \in E^{n+1} \mid \delta \in \Delta_\gamma,\ \zeta \leqslant g'(\gamma, \delta)\},$$

$$\begin{aligned}[\varphi, L^*_x]^*_y = [g', \Lambda'] &= \bigcup_{\gamma\in\Gamma'} [\varphi_\gamma, D_\gamma]^* \\ &= \{(\zeta, \gamma, \delta) \in E^{m+n+1} \mid (\gamma, \delta) \in \Lambda',\ \zeta \leqslant g'(\gamma, \delta)\}. \end{aligned} \tag{6}$$

Finally, denoting by Δ'' the projection of the set L^*_y onto E^n and fixing $\delta \in \Delta''$, put

$$C_\delta = \{x \in E^m \mid (x, \delta) \in L^*_y\},$$

$$[\psi_\delta, C_\delta] = \{(\beta, x) \in E^{m+1} \mid x \in C_\delta,\ \beta \leqslant \psi(x, \delta)\}.$$

As before, define

$$\Gamma_\delta = \left\{\gamma \in E^m \,\middle|\, \sup_{x\in C_\delta} [\psi(x, \delta) + \gamma x] < \infty\right\},$$

$$\Lambda'' = \bigcup_{\delta \in \Delta''} \Gamma_\delta = \{(\gamma, \delta) \in E^{m+n} \mid \delta \in \Delta'',\ \gamma \in \Gamma_\delta\},$$

$$g''(\gamma, \delta) = \sup_{x \in C_\delta} [\psi(x, \delta) + \gamma x], \tag{7}$$

$$[\psi_\delta, C_\delta]^* = \{(\zeta, \gamma) \in E^{m+1} \mid \gamma \in \Gamma_\delta,\ \zeta \geqslant g''(\gamma, \delta)\},$$

$$\begin{aligned} [\psi, L_y^*]_x^* = [g'', \Lambda''] &= \bigcup_{\delta \in \Delta''} [\psi_\delta, C_\delta]^* \\ &= \{(\zeta, \gamma, \delta) \in E^{m+n+1} \mid (\gamma, \delta) \in \Lambda'',\ \zeta \geqslant g''(\gamma, \delta)\}. \end{aligned} \tag{8}$$

By virtue of the properties of the set L and of the function $f(x, y)$, the following assertions hold.

Lemma 1. The set L_x^* (L_y^*) is convex and nonempty, and the function $\varphi(\psi)$, defined on L_x^* (L_y^*), is convex (concave) and closed.

Lemma 2. $[[f^-, L]_{x^*}]_{y^*} = [f^-, L]$ and $[[f^+, L]_{y^*}]_{\delta^*} = [f^+, L]$.

Lemma 3. The set Λ' is convex and nonempty, moreover $\Lambda' = \Gamma' \times \Delta'$, where Δ' is the projection of the set Λ' onto E^n, and the function $g'(\gamma, \delta)$, defined on Λ', is closed, convex in γ, and concave in δ.

The analogous assertion is true for Λ'' and g''.

Lemma 4. $[[\varphi, L_x^*]_{y^*}]_{\delta^*} = [\varphi, L_x^*]$ and $[[\psi, L_y^*]_{x^*}]_{\gamma^*} = [\psi, L_y^*]$.

Lemma 5. $\Lambda' = \Lambda'' = \Lambda$, $g'(\gamma, \delta) = g''(\gamma, \delta) = g(\gamma, \delta)$.

From Lemmas 2, 4, and 5 it follows that the set Λ and the function g may be defined directly, namely

$$\begin{aligned} \Lambda = \{(\gamma, \delta) \in E^{m+n} \mid\ -\infty &< \inf_{y \in D} \sup_{x \in C} [f(x, y) + \gamma x - \delta y] \\ &= \sup_{x \in C} \inf_{y \in D} [f(x, y) + \gamma x - \delta y] < \infty\}, \end{aligned} \tag{9}$$

$$g(\gamma, \delta) = \inf_{y \in D} \sup_{x \in C} [f(x, y) + \gamma x - \delta y] = \sup_{x \in C} \inf_{y \in D} [f(x, y) + \gamma x - \delta y]. \tag{10}$$

It is natural to denote $[f^-, L]^* = [g^-, \Lambda]$, $[f^+, L]^* = [g^+, \Lambda]$.

Corollary 1. For any $\gamma \in \Gamma,\ \delta \in \Delta$,

$$\inf_{y \in D} \sup_{x \in C} [f(x, y) + \gamma x - \delta y] = \sup_{x \in C} \inf_{y \in D} [f(x, y) + \gamma x - \delta y].$$

Corollary 2. If the sets C, D are compact, then

$$\min_{y\in D}\max_{x\in C} f(x,y) = \max_{x\in C}\min_{y\in D} f(x,y).$$

From the definitions and lemmas there follows the validity of the following theorems.

Theorem 1. The sets $[\varphi, L_x^*]$, $[\psi, L_y^*]$ are conjugate.

Theorem 2. The set Λ is convex and nonempty, moreover $\Lambda = \Gamma \times \Delta$, and the function $g(\gamma,\delta)$, defined on Λ, is closed, convex in γ, and concave in δ.

2. Duality theorems. Let C_1 and C_2 be convex sets in E^m; D_1 and D_2 convex sets in E^n; $x \in E^m$ and $y \in E^n$. The function $f_1(x,y)$ is defined on $L_1 = C_1 \times D_1$, and the function $f_2(x,y)$ is defined on $L_2 = C_2 \times D_2$. The functions f_1 and f_2 are concave in x, convex in y, and closed on L_1 and L_2, respectively.

Denote

$$[f_1^-, L_1]_{x^*} = [\varphi_1, L_{1x}^*], \qquad [f_2^-, L_2]_{x^*} = [\varphi_2, L_{2x}^*],$$

$$[f_1^+, L_1]_{y^*} = [\psi_1, L_{1y}^*], \qquad [f_2^+, L_2]_{y^*} = [\psi_2, L_{2y}^*],$$

$$[f_1^-, L_1]^* = [g_1^-, \Lambda_1], \qquad [f_2^-, L_2]^* = [g_2^-, \Lambda_2],$$

$$[f_1^+, L_1]^* = [g_1^+, \Lambda_1], \qquad [f_2^+, L_2]^* = [g_2^+, \Lambda_2],$$

where $\Lambda_1 = \Gamma_1 \times \Delta_1$ and $\Lambda_2 = \Gamma_2 \times \Delta_2$.

Put

$$-\Gamma_y = \{\gamma \in E^m \mid \sup_{x\in C}[f(x,y) - \gamma x] < \infty\},$$

$$-\Delta_x = \{\delta \in E^n \mid \inf_{y\in D}[f(x,y) + \delta y] > -\infty\},$$

$$L_x^* = \bigcup_{y\in D}(-\Gamma_y), \qquad L_y^* = \bigcup_{x\in C}(-\Delta_x),$$

$$-\Lambda = \{\{\gamma,\delta\} \in E^{m+n} \mid -\infty < \inf_{y\in D}\sup_{x\in C}[f(x,y) - \gamma x + \delta y] =$$

$$= \sup_{x \in C} \inf_{y \in D} [f(x, y) - \gamma x + \delta y] < \infty\}.$$

We shall denote the projections of the set $-\Lambda$ onto E^m and E^n, respectively, by $-\Gamma$ and $-\Delta$.

Obviously, by virtue of definitions (1)—(4), (9), and (10), on the sets L_x^*, L_y^* and $-\Lambda$ the corresponding functions $\varphi(-\gamma, y)$, $\psi(x, -\delta)$, and $g(-\gamma, -\delta)$ are defined.

Let us formulate the following problems:

Problem 1. Find

$$\min_{y \in D_1 \cap D_2} \sup_{x \in C_1 \cap C_2} [f_1(x, y) + f_2(x, y)].$$

Problem 1′**.** Find

$$\max_{x \in C_1 \cap C_2} \inf_{y \in D_1 \cap D_2} [f_1(x, y) + f_2(x, y)].$$

Problem 2. Find

$$\min_{(\gamma, y) \in L_{1x}^* \cap L_{2x}^{*'}} [\varphi_1(\gamma, y) + \varphi_2(-\gamma, y)].$$

Problem 2′**.** Find

$$\max_{(x, \delta) \in L_{1y}^* \cap L_{2y}^{*'}} [\psi_1(x, \delta) + \psi_2(x, -\delta)].$$

Problem 3. Find

$$\min_{\gamma \in \Gamma_1 \cap (-\Gamma_2)} \sup_{\delta \in \Delta_1 \cap (-\Delta_2)} [g_1(\gamma, \delta) + g_2(-\gamma, -\delta)].$$

Problem 3′**.** Find

$$\max_{\delta \in \Delta_1 \cap (-\Delta_2)} \inf_{\gamma \in \Gamma_1 \cap (-\Gamma_2)} [g_1(\gamma, \delta) + g_2(-\gamma, -\delta)].$$

Existence theorem. If the sets $L_{1x}^* \cap L_{2x}^{*'}$ and $L_{1y}^* \cap L_{2y}^{*'}$ are nonempty, then:

$$
\begin{aligned}
\inf_{y\in D_1\cap D_2}\ \sup_{x\in C_1\cap C_2}[f_1(x,y)+f_2(x,y)] &= \sup_{x\in C_1\cap C_2}\ \inf_{y\in D_1\cap D_2}[f_1(x,y)+f_2(x,y)]\\
&= \inf_{(\gamma,y)\in L^*_{1x}\cap L^{*'}_{2x}}[\varphi_1(\gamma,y)+\varphi_2(-\gamma,y)]\\
&= \sup_{(x,\delta)\in L^*_{1y}\cap L^{*'}_{2y}}[\psi_1(x,\delta)+\psi_2(x,-\delta)]\\
&= \inf_{\gamma\in\Gamma_1\cap(-\Gamma_2)}\ \sup_{\delta\in\Delta_1\cap(-\Delta_2)}[g_1(\gamma,\delta)+g_2(-\gamma,-\delta)]\\
&= \sup_{\delta\in\Delta_1\cap\Delta_2}\ \inf_{\gamma\in\Gamma_1\cap(-\Gamma_2)}[g_1(\gamma,\delta)+g_2(-\gamma,-\delta)].
\end{aligned}
$$

The validity of this theorem follows from Theorems 1 and 2 and the existence theorem of convex programming (items 7, 8 (1) and (2)).

The existence theorem does not give solvability conditions for problems $1, 1', 2, 2', 3$, and $3'$. This shortcoming is remedied by the duality theorems and their corollaries given below.

First duality theorem. If problems 2 and $2'$ are solvable, then problems $1, 1', 3$, and $3'$ are also solvable, and the optimal values are equal.

The validity of this theorem follows directly from the existence theorem.

Corollary 1. If the sets $L^*_{1x}\cap L^{*'}_{2x}$, $L^*_{1y}\cap L^{*'}_{2y}$ are compact, then all the problems are solvable.

Corollary 2. If the set $L^*_{1x}\cap L^{*'}_{2x}$ $(L^*_{1y}\cap L^{*'}_{2y})$ has a relative interior point and problem 2 $(2')$ is solvable, then all the problems are solvable.

Corollary 3. If the sets $L^*_{1x}\cap L^{*'}_{2x}$, $L^*_{1y}\cap L^{*'}_{2y}$ have relative interior points, then all the problems are solvable.

Second duality theorem. If the sets $C_1\cap C_2$, $D_1\cap D_2$ have relative interior points and problems 1 and $1'$ are solvable, then problems $2, 2', 3$, and $3'$ are also solvable, and the optimal values are equal.

The proof is based on the existence theorem given above and the duality theorem of convex programming (2).

Corollary. If the sets $C_1\cap C_2$, $D_1\cap D_2$ $(\Gamma_1\cap(-\Gamma_2), \Delta_1\cap(-\Delta_2))$ are compact and have relative interior points, then all the problems are solvable.

3. Applications to concave-convex games. If we put $C_1 = C_2 = C$, $D_1 = D_2 = D$, and $f_1(x,y)+f_2(x,y) = f(x,y)$, then problems 1 and $1'$ can be interpreted as the minimax and maximin problems, respectively, for the concave-convex game $\{C, D, f\}$ with kernel $f(x,y)$. The theory set forth shows the possibility of reducing such games to conjugate problems of convex programming (problems 2 and $2'$) and conversely.

To the game $\{C, D, f\}$ there corresponds a conjugate concave-convex game, determined by the triple $\{\Delta_1 \cap (-\Delta_2), \Gamma_1 \cap (-\Gamma_2), g_1 + g_2\}$ with kernel $g_1(\gamma, \delta) + g_2(-\gamma, -\delta)$. The solvability conditions for the original and conjugate games and for the corresponding conjugate problems of convex programming are given by the duality theorems and their corollaries presented above.

Received
23 VIII 1966

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