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Abstract

Full Text

MATHEMATICS

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A NECESSARY AND SUFFICIENT CONDITION FOR THE RIEMANN INTEGRABILITY OF CONSTRUCTIVE FUNCTIONS

(Presented by Academician M. A. Lavrent' ev on 21 XII 1966)

The definition of Riemann and Lebesgue integrability of constructive functions on the segment $0\Delta 1$ is given in papers ([1,2]). In the present note we shall use the terms and notation introduced in ([2]). A **regular covering** will mean such a covering ([2]) for which the sum of the lengths of the segments converges to 1. For a function f, $R(f)$ (respectively $L(f)$) will denote: f is Riemann integrable (Lebesgue integrable) on the segment $0\Delta 1$.

Theorem. *A function f is Riemann integrable on $0\Delta 1$ if and only if there exist a natural number K and a sequence of natural numbers $\{k_n\}$ for which the following hold:*

1) $\forall x(|f(x)| \leqslant K)$;
2) *for every natural n and for every finite system of nonoverlapping segments $\{a_i \Delta b_i\}_{i=1}^{s}$ such that*

$$\forall i \left(1 \leqslant i \leqslant s \supset 0 \leqslant a_i < b_i \leqslant 1 \& (b_i - a_i) < \frac{1}{k_n + 1} \;\&\; \exists cd \left(a_i \leqslant c < d \leqslant b_i \& |f(c) - f(d)| > \frac{1}{n+1}\right)\right),$$

one has

$$\sum_{i=1}^{s} (b_i - a_i) < \frac{1}{n+1}.$$

With the aid of this theorem one can easily prove the following assertions:

Corollary 1. *Let f be a Riemann integrable function. Then f is Lebesgue integrable, and the value of the Lebesgue integral of f is equal to the value of the Riemann integral of f.*

Corollary 2. *Let f and g be Riemann integrable functions, c a positive rational number, and Φ a covering. Then $R(f+g)$, $R(f \cdot g)$, $R(|f|^c)$, $R(f/\Phi)$, and if for some natural m it is fulfilled that $\forall x \left(|f(x)| \geqslant \frac{1}{m+1}\right)$, then $R\left(\frac{1}{f}\right)$.*

Corollary 3. *Let f be a function such that $R(|f|)$. Then $R(f)$.*

Remark. One can construct a function f, bounded functions g and h, and a covering Φ such that the functions f, $|g|$, and h are Lebesgue integrable on $0\Delta1$, but

$$\forall c\,(1 < c \supset \neg L(|f|^c))\ \&\ \neg L(g)\ \&\ \neg L(h/\Phi)$$

holds, and, consequently, the functions f, $|g|$, and h are not Riemann integrable.

In conclusion we give two sufficient conditions for Riemann integrability of functions.

Corollary 4. Let f and g be functions, and suppose

$$R(f)\ \&\ \forall ab\,(|g(a) - g(b)| \leq |f(a) - f(b)|).$$

Then $R(g)$.

Corollary 5. Let f be a bounded function, and suppose that for every natural n there exists a system of nonoverlapping segments

$$\{\,{}^n c_i \Delta {}^n d_i\,\}_{i=1}^{l_n}$$

such that

$$\sum_{i=1}^{l_n} ({}^n d_i - {}^n c_i) > 1 - \frac{1}{n+1},$$

and for every natural i from 1 to l_n one has $0 \leq {}^n c_i < {}^n d_i \leq 1$, and f is uniformly continuous on the segment ${}^n c_i \Delta {}^n d_i$. Then f is Riemann integrable.

Remark. On the basis of the last corollary we immediately obtain that every bounded function f is Riemann integrable for which there exists a regular covering Φ such that

$$\forall x\,(f(x) = (f/\Phi)(x)).$$

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REFERENCES

[1] B. A. Kushner, DAN, 156, No. 2 (1964). [2] O. Demuth, DAN, 160, No. 6 (1965).

Note: Figure translations are in progress. See original paper for figures.