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HYDROMECHANICS

1967

SovietRxiv

Abstract

Full Text

UDC 534.222.2+517.946

HYDROMECHANICS

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ON THE EQUATIONS OF SHORT WAVES IN A VISCOUS HEAT-CONDUCTING GAS

(Presented by Academician L. I. Sedov, 12 IV 1966)

The paper gives a derivation of asymptotic equations for flows of the short-wave type, where rather abrupt changes in the flow parameters occur in narrow regions adjacent to shock fronts ([1,2]), and their simplification in various particular cases of motions of a viscous heat-conducting gas. Some of the equations obtained had, until recently, not been encountered in practice.

1. General equations of short waves for a viscous heat-conducting gas were obtained in ([3]). The equations of continuity, Navier–Stokes, conservation of energy, and also the equation of state for two-dimensional unsteady motions may be represented in the form ([4–6]):

$$\partial\rho/\partial t + \partial(\rho v_x)/\partial x + \partial(\rho v_y)/\partial y + (k-1)\rho v_y/y = 0; \tag{1,1}$$

$$\rho\left(\frac{\partial v_x}{\partial t} + v_x\frac{\partial v_x}{\partial x} + v_y\frac{\partial v_x}{\partial y}\right) = -\frac{\partial p}{\partial x} + \frac{\partial}{\partial x}\left\{2\lambda\frac{\partial v_x}{\partial x} + \left(\zeta - \frac{2}{3}\lambda\right)\left[\frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} + \frac{(k-1)v_y}{y}\right]\right\}$$
$$+\frac{\partial}{\partial y}\left[\lambda\left(\frac{\partial v_x}{\partial y} + \frac{\partial v_y}{\partial x}\right)\right] + \frac{(k-1)\lambda}{y}\left(\frac{\partial v_x}{\partial y} + \frac{\partial v_y}{\partial x}\right); \tag{1,2}$$

$$\rho\left(\frac{\partial v_y}{\partial t} + v_x\frac{\partial v_y}{\partial x} + v_y\frac{\partial v_y}{\partial y}\right) = -\frac{\partial p}{\partial y} + \frac{\partial}{\partial x}\left[\lambda\left(\frac{\partial v_x}{\partial y} + \frac{\partial v_y}{\partial x}\right)\right]$$
$$+\frac{\partial}{\partial y}\left\{2\lambda\frac{\partial v_y}{\partial y} + \left(\zeta - \frac{2}{3}\lambda\right)\left[\frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} + \frac{(k-1)v_y}{y}\right]\right\} + \frac{2(k-1)\lambda}{y}\left(\frac{\partial v_y}{\partial y} - \frac{v_y}{y}\right); \tag{1,3}$$

$$\rho c_p\left(\frac{\partial T}{\partial t} + v_x\frac{\partial T}{\partial x} + v_y\frac{\partial T}{\partial y}\right) - \alpha T\left(\frac{\partial p}{\partial t} + v_x\frac{\partial p}{\partial x} + v_y\frac{\partial p}{\partial y}\right) =$$
$$= \frac{\partial}{\partial x}\left(\chi\frac{\partial T}{\partial x}\right) + \frac{\partial}{\partial y}\left(\chi\frac{\partial T}{\partial y}\right) + \frac{(k-1)\chi}{y}\frac{\partial T}{\partial y} + 2\lambda\left\{\left(\frac{\partial v_x}{\partial x}\right)^2\right.$$

$$+\frac{1}{2}\left(\frac{\partial v_x}{\partial y}+\frac{\partial v_y}{\partial x}\right)^2+\left(\frac{\partial v_y}{\partial y}\right)^2+\left[\frac{(k-1)v_y}{y}\right]^2\Bigg\}$$

$$+\left(\zeta-\frac{2}{3}\lambda\right)\left[\frac{\partial v_x}{\partial x}+\frac{\partial v_y}{\partial y}+\frac{(k-1)v_y}{y}\right]^2; \qquad (1,4)$$

$$\alpha\rho a^2\,dT=\gamma\,dp-a^2\,d\rho \qquad \left(a^2\alpha^2T=(\gamma-1)c_p\right). \qquad (1,5)$$

Here t is time; x and y are orthogonal Cartesian coordinates (for plane-parallel flows) and cylindrical coordinates (for axisymmetric flows); v_x and v_y are the corresponding components of the velocity vector; ρ is density; p is pressure; T is temperature; λ, ζ, χ are the coefficients of viscosity, second viscosity, and thermal conductivity; a is the adiabatic speed of sound; α is the coefficient of thermal expansion; γ is the ratio of the heat capacity at constant pressure c_p to the heat capacity at constant volume c_v. For plane-parallel flows $k=1$; for axisymmetric flows $k=2$.

The specific entropy s can then be found from equation (6)

$$\rho T\,ds=\rho c_p\,dT-\alpha T\,dp.$$

Suppose that a wave is propagating in the direction of the x-axis through an unperturbed gas at rest with parameters $p_0, \rho_0, T_0, \ldots$, in which the excess values of all quantities are small in comparison with the initial ones. We pass to dimensionless variables in a moving coordinate system

$$\frac{x}{a_0t}=1+L\xi, \qquad \frac{y}{a_0t}=\frac{L}{\varepsilon}\eta, \qquad t=\frac{\mu}{\rho_0a_0^2}\frac{\tau}{\Delta}, \qquad v_x=a_0Mu, \quad v_y=a_0M\varepsilon v, \qquad (1,6)$$

$$p=\rho_0a_0^2(p_0/\rho_0a_0^2+MP), \qquad \rho=\rho_0(1+MR),$$

$$T=T_0(1+M\theta), \qquad a=a_0(1+MA).$$

Here $\xi, \eta, \tau, u, v, P, R, \theta, A$ are quantities of order unity, while $L, \varepsilon, \Delta, M$ are small in comparison with it, and $\mu={}^4/_3\lambda+\zeta$. The quantities $\lambda, \zeta, \varkappa, c_p$ and c_v may be regarded as constant. Taking their perturbations into account only somewhat complicates the calculations, but does not change the final approximate equations [(3)].

Let us substitute the variables $(1,6)$ into equations $(1,1)$–$(1,5)$ and retain in the resulting relations only the leading terms. We shall assume [(4)] that the coefficients of viscosity and thermal conductivity have the same order of magnitude and are sufficiently small that the condition $\Delta \ll L$ is satisfied. Depending on the order of magnitude of the time derivatives $\partial/\partial\tau$, different equations will be obtained. Assuming at once that $\partial/\partial\tau \ll 1/L$, we thereby exclude the ordinary linearized equations of motion of a viscous heat-conducting gas. The general equations of short waves take the form

$$P = u, \qquad R = u, \qquad \theta = [(\gamma - 1)/\alpha_0 T_0]u; \tag{1,7}$$

$$\partial v/\partial\xi = \partial u/\partial\eta;$$

$$\tau\frac{\partial u}{\partial\tau} + \left(\frac{M}{L}m_0 u - \xi\right)\frac{\partial u}{\partial\xi} - \eta\frac{\partial u}{\partial\eta} + \frac{1}{2}\frac{\varepsilon^2}{L}\left[\frac{\partial v}{\partial\eta} + \frac{(k-1)v}{\eta}\right] - \frac{\Delta}{2L^2\tau}\left(1 + \frac{\gamma - 1}{\mathrm{Pr}}\right)\frac{\partial^2 u}{\partial\xi^2} = 0. \tag{1,8}$$

Here m_0 denotes the value of the quantity $m = {}^1/_2\rho^{-3}a^{-2}[\partial^2 p/\partial(1/\rho)^2]_s$ in the unperturbed gas; for a perfect gas (obeying the Clapeyron equation of state) $m = {}^1/_2(\gamma + 1)$.

We note that, although the coefficients of viscosity and thermal conductivity have not entered explicitly into these equations, their order is taken into account by the quantity Δ, while their ratio is determined by the Prandtl number $\mathrm{Pr} = \mu c_p/\varkappa$.

The relations $(1, 7)$, obtained respectively from equations $(1, 1)$, $(1, 2)$, and $(1, 5)$, are analogous to the connections between perturbations of the gas parameters in a plane traveling pulse of small amplitude. They mean that, in the approximation under consideration, compression of the gas occurs adiabatically. The first equation $(1, 8)$, following from $(1, 3)$ and $(1, 7)$, is the condition that the flow be irrotational. The second equation $(1, 8)$ is obtained from the energy conservation equation $(1, 4)$, if the quantities of the first order of smallness connected with the transfer of mass of the substance and its momentum are excluded from it.

After the change of variables

$$\xi' = \xi, \qquad \eta' = \sqrt{2}\eta, \qquad \tau' = 2[1+(\gamma-1)/\mathrm{Pr}]^{-1}\tau, \qquad u' = m_0 u, \qquad v' = (m_0/\sqrt{2})v$$

equations $(1, 8)$ take the form (we omit the primes over the variables)

$$\partial v/\partial\xi = \partial u/\partial\eta, \tag{1,9}$$

$$\tau\frac{\partial u}{\partial\tau} + \left(\frac{M}{L}u - \xi\right)\frac{\partial u}{\partial\xi} - \eta\frac{\partial u}{\partial\eta} + \frac{\varepsilon^2}{L}\left[\frac{\partial v}{\partial\eta} + \frac{(k-1)v}{\eta}\right] - \frac{\Delta}{L^2\tau}\frac{\partial^2 u}{\partial\xi^2} = 0. \tag{1,10}$$

From these equations follows the equation for the potential of the perturbed velocity $\Phi(\tau, \xi, \eta)$

$$\tau\frac{\partial^2\Phi}{\partial\tau\partial\xi}+\left(\frac{M}{L}\frac{\partial\Phi}{\partial\xi}-\xi\right)\frac{\partial^2\Phi}{\partial\xi^2}-\eta\frac{\partial^2\Phi}{\partial\xi\partial\eta}+\frac{\varepsilon^2}{L}\left(\frac{\partial^2\Phi}{\partial\eta^2}+\frac{k-1}{\eta}\frac{\partial\Phi}{\partial\eta}\right)-\frac{\Delta}{L^2\tau}\frac{\partial^3\Phi}{\partial\xi^3}=0. \tag{1,11}$$

Equations (1,9), (1,10), or the equivalent equation (1,11), are the equations of short waves in a viscous heat-conducting gas and serve as the basis for the subsequent investigation.

2. In different problems the terms entering equation (1,10) may have different orders of magnitude. Below we shall give the short-wave equations, simplified in special cases, which follow from (1,10). We note that to the consequences of this equation one must each time add the condition of irrotationality (1,9).

If in equation (1,10) one may neglect the last term, representing the influence of dissipative factors, then one obtains the equations of short waves for an ideal gas ([1,2]). We shall not dwell on these cases.

Let us first suppose that the order of the derivatives with respect to time of the dimensionless flow parameters is comparable with unity, $\partial/\partial\tau \sim 1$. Then, in the case $M = L,\ \varepsilon = L^{1/2},\ \Delta = L^2$, all the terms of equation (1,10) are of the same order of magnitude, and it takes the form

$$\tau\frac{\partial u}{\partial\tau}+(u-\xi)\frac{\partial u}{\partial\xi}-\eta\frac{\partial u}{\partial\eta}+\frac{\partial v}{\partial\eta}+\frac{(k-1)v}{\eta}-\frac{1}{\tau}\frac{\partial^2 u}{\partial\xi^2}=0. \tag{2,1}$$

This nonlinear equation, obtained in work ([3]), together with (1,9), determines, for example, the two-dimensional structure of the flow in a neighborhood of a triple point for the problem of Mach reflection of a weak shock wave.

Very weak short waves ($M \ll L$) in the general case $\varepsilon = L^{1/2},\ \Delta = L^2$ are described by the linear equation

$$\tau\frac{\partial u}{\partial\tau}-\xi\frac{\partial u}{\partial\xi}-\eta\frac{\partial u}{\partial\eta}+\frac{\partial v}{\partial\eta}+\frac{(k-1)v}{\eta}-\frac{1}{\tau}\frac{\partial^2 u}{\partial\xi^2}=0. \tag{2,2}$$

If the motion of the wave along each ray does not depend on the motion along neighboring rays, i.e., is quasi-one-dimensional, and the transverse motion may be neglected: $\varepsilon \ll L^{1/2},\ M = L,\ \Delta = L^2$, then one obtains the parabolic equation

$$\tau\frac{\partial u}{\partial\tau}+(u-\xi)\frac{\partial u}{\partial\xi}-\eta\frac{\partial u}{\partial\eta}-\frac{1}{\tau}\frac{\partial^2 u}{\partial\xi^2}=0, \tag{2,3}$$

and for very weak waves ($M \ll L$) this equation proves to be linear

$$\tau\frac{\partial u}{\partial \tau} - \xi\frac{\partial u}{\partial \xi} - \eta\frac{\partial u}{\partial \eta} - \frac{1}{\tau}\frac{\partial^2 u}{\partial \xi^2} = 0. \tag{2,4}$$

If the wavelength, in order of magnitude, is considerably smaller than its amplitude ($L \ll M$), then such waves are called very short, and their general equation ($M = \varepsilon^2 = \Delta/L \gg L$) has the form

$$u\frac{\partial u}{\partial \xi} + \frac{\partial v}{\partial \eta} + \frac{(k-1)v}{\eta} - \frac{1}{\tau}\frac{\partial^2 u}{\partial \xi^2} = 0. \tag{2,5}$$

The derivative with respect to time has not entered this equation, which indicates the quasi-steady character of the corresponding motion of the gas. Such an equation was apparently first obtained by Lighthill, Ashkenas, and Cole ([7]) for the description of two-dimensional stationary transonic flows of a perfect gas. Equation (2,5) was considered in work ([3]), where it was shown that the presence of viscosity and heat conduction changes the asymptotic pattern of the stationary flow past bodies of revolution by a stream that is sonic at infinity.

Very short quasi-one-dimensional waves ($M = \Delta/L \gg L, \varepsilon^2$) obey the equation

$$u\frac{\partial u}{\partial \xi} - \frac{1}{\tau}\frac{\partial^2 u}{\partial \xi^2} = 0, \tag{2,6}$$

which coincides with the equation for the structure of shock waves and is readily integrated.

Of great interest is the equation

$$\frac{\partial v}{\partial \eta} + \frac{(k-1)v}{\eta} - \frac{1}{\tau}\frac{\partial^2 u}{\partial \xi^2} = 0, \tag{2,7}$$

describing short waves in which only the transverse flow and dissipative factors are significant. The corresponding equation for the velocity potential has the form

$$\frac{\partial^2 \Phi}{\partial \eta^2} + \frac{k-1}{\eta}\frac{\partial \Phi}{\partial \eta} - \frac{1}{\tau}\frac{\partial^3 \Phi}{\partial \xi^3} = 0. \tag{2,7a}$$

The latter equation, which is of parabolic type, had until recently not occurred in physical problems. An analogous equation was first considered in work [8] for describing the flow of a viscous heat-conducting gas far from a body of revolution in a sonic stream. Earlier, certain general properties of the operator $\partial^2/\partial y^2 + \partial^3/\partial x^3$ were investigated in works [9,10].

In deriving equations (2,1)–(2,7) it was assumed that $\partial/\partial\tau \sim 1$. If, however, the change of the wave parameters with time occurs more slowly ($\partial/\partial\tau \ll 1$), then in equation (1,10) the first term may immediately be omitted. This will correspond to quasi-stationary, and in the original variables—to quasi-automodel flows. In particular, very weak waves in this case satisfy the equation

$$\xi\frac{\partial u}{\partial \xi} + \eta\frac{\partial u}{\partial \eta} - \frac{\partial v}{\partial \eta} - \frac{(k-1)v}{\eta} + \frac{1}{\tau}\frac{\partial^2 u}{\partial \xi^2} = 0, \tag{2,8}$$

which is simplified still further if the transverse motion of the gas may be neglected,

$$\xi\frac{\partial u}{\partial \xi} + \eta\frac{\partial u}{\partial \eta} + \frac{1}{\tau}\frac{\partial^2 u}{\partial \xi^2} = 0. \tag{2,9}$$

Short waves with a more rapid change of parameters in time ($1 \ll \partial/\partial\tau \ll 1/L$), as follows from (1,10), are described in the general case ($\partial/\partial\tau \sim M/L = \varepsilon^2/L = \Delta/L^2 \gg 1$) by the equation

$$\tau\frac{\partial u}{\partial \tau} + u\frac{\partial u}{\partial \xi} + \frac{\partial v}{\partial \eta} + \frac{(k-1)v}{\eta} - \frac{1}{\tau}\frac{\partial^2 u}{\partial \xi^2} = 0. \tag{2,10}$$

If we now neglect the nonlinear term, the velocity potential satisfies the equation

$$\tau\frac{\partial^2 \Phi}{\partial \tau\,\partial \xi} + \frac{\partial^2 \Phi}{\partial \eta^2} + \frac{k-1}{\eta}\frac{\partial \Phi}{\partial \eta} - \frac{1}{\tau}\frac{\partial^3 \Phi}{\partial \xi^3} = 0, \tag{2,11}$$

which differs from (2,7a) only by the first term. For quasi-one-dimensional motions in this case one again obtains the heat-conduction equation

$$\tau\frac{\partial u}{\partial \tau} - \frac{1}{\tau}\frac{\partial^2 u}{\partial \xi^2} = 0. \tag{2,12}$$

The author thanks O. S. Ryzhov for discussion of the results of the work.

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Received
11 IV 1966

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Note: Figure translations are in progress. See original paper for figures.