LIMITS OF UNCERTAINTY IN THE SENSE OF T-MEANS FOR TRIGONOMETRIC SERIES

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Abstract

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MATHEMATICS

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Let us take an infinite matrix

$$T = \|a_{mk}\| \qquad (m, k = 0, 1, 2, ...) \tag{1}$$

and some infinite series

$$\sum_{n=0}^{\infty} u_n \tag{2}$$

with partial sums S_m $(m = 0, 1, 2, ...)$. We shall call the quantities

$$A_m(T) = \sum_{k=0}^{\infty} a_{mk} S_k \qquad (m = 0; 1, 2, ...) \tag{3}$$

the T-means of the series (2), determined by the matrix (1).

It is said that the series (2) is summable by the method T to the value S, if the series on the right-hand side of equality (3) converge for every $m = 0, 1, 2, ...$ and the quantities $A_m(T)$ tend to the limit S as $m \to \infty$. The summation method T is called regular if every series (2) converging to a finite value S is summable by the method T to the same value S.

The summation method T is called a method with finite rows if the matrix (1) defining this method satisfies the condition

$$a_{mk} = 0 \qquad (k > \nu_m,\ m = 0, 1, 2, ...), \tag{4}$$

where ν_m are natural numbers, in general depending on m.

Theorem 1. *Let an arbitrary regular summation method T with finite rows be given, determined by the matrix (1). Then, for any two measurable functions $F(x)$ and $G(x)$ satisfying the inequality*

$$G(x) \leq F(x) \tag{5}$$

almost everywhere on the interval $[-\pi, \pi]$, *one can define a trigonometric series*

$$\frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx), \tag{6}$$

which has the following properties:

1) If

$$A_m(x, T) \qquad (m = 0, 1, 2, ...) \tag{7}$$

*are the T-means of the series (6), determined by the matrix (1), then the upper limit in measure on $[-\pi, \pi]$ of the sequence (7) is equal to $F(x)$, and the lower limit in measure on $[-\pi, \pi]$ of the same sequence is equal to $G(x)$**.*

2)

$$\lim_{n\to\infty} a_n = 0, \qquad \lim_{n\to\infty} b_n = 0. \tag{8}$$

* The functions $F(x)$ and $G(x)$ may be equal to $+\infty$ or $-\infty$ on sets of positive measure.

** The definition of upper and lower limits in measure of a sequence of functions is given in (1), p. 4.

If in Theorem 1 we assume that $F(x) = G(x)$ almost everywhere on $[-\pi, \pi]$, then we obtain the following theorem.

Theorem 2. *For any measurable function $F(x)$, defined almost everywhere on the segment $[-\pi, \pi]$, one can define a trigonometric series (6), satisfying condition (8), which converges in measure on $[-\pi, \pi]$ to the function $F(x)$.*

In the proof of Theorem 1 the following is used.

Lemma A. *Let there be given: an arbitrary regular summability method T with finite rows, defined by the matrix (1); an arbitrary function $\varphi(x)$, continuous on the segment $[-\pi, \pi]$; an arbitrary positive number $\sigma < 1$, and arbitrary natural numbers L and L_0.*

Then one can define a natural number $L' > L$, a trigonometric polynomial

$$H(x) = \sum_{j=L+1}^{L'} (a_j \cos jx + b_j \sin jx)$$

and sets E, G_n $(L < n \leq L')$, which satisfy the following conditions:

1) $\operatorname{mes} E < \sigma,\ E \subset [-\pi, \pi]$;

2) $\operatorname{mes} G_n < \sigma,\ G_n \subset [-\pi, \pi]\ (L < n \leq L')$;

3) if $\tau_k(x)$ and $B_n(x, T)$ are determined from the equalities

$$\tau_k(x) = \begin{cases} 0 & (0 \leq k \leq L), \\ \sum_{j=L+1}^{k} (a_j \cos jx + b_j \sin jx) & (L < k \leq L'), \\ H(x) & (k > L'), \end{cases}$$

$$B_n(x, T) = \sum_{k=0}^{\infty} a_{nk} \tau_k(x) \qquad (n = 0, 1, 2, ...),$$

then

$$|B_n(x, T) - \varphi(x)| < \sigma \qquad (x \in [-\pi, \pi] - E,\ n \geq L');$$

4) $B_n(x, T) = \theta_n(x)\varphi(x) + \eta_n(x)\ (x \in [-\pi, \pi] - G_n,\ L < n \leq L')$,

where $|\theta_n(x)| < K,\ |\eta_n(x)| < \sigma\ (x \in [-\pi, \pi] - G_n,\ L < n \leq L')$, and K is a constant depending only on the summability method T;

5) $B_n(x, T) = 0\ (0 \leq n \leq L)$;

6) $L_0 < L',\ \sqrt{a_n^2 + b_n^2} < \sigma\ (L < n \leq L')$.

Definition of a trigonometric series (6) satisfying the conditions of Theorem 1. Let us take an arbitrary summability method T and arbitrary functions $F(x), G(x)$ satisfying the conditions of Theorem 1, and put

$$f_{2\nu}(x) = F(x), \qquad f_{2\nu+1}(x) = G(x) \qquad (\nu = 0, 1, 2, ...).$$

Take a sequence of functions $h_m(x)$ $(m = 0, 1, 2, ...)$, continuous on $[-\pi, \pi]$, which is an almost uniformly convergent sequence to the sequence of functions $f_m(x)$ $(m = 0, 1, 2, ...)$ almost everywhere on $[-\pi, \pi]$. Put, further,

$$u_m(x) = \max[h_m(x), h_{m+1}(x)], \qquad v_m(x) = \min[h_m(x), h_{m+1}(x)]$$

$$(m = 0, 1, 2, ...),$$

$$w_{2\nu}(x) = v_\nu(x), \qquad w_{2\nu+1}(x) = u_\nu(x), \qquad (\nu = 0, 1, 2, ...),$$

$$\mu_m = [2^m \Omega_m] = 1 \qquad (m = 1, 2, ...),$$

* If the function $F(x)$ is finite almost everywhere on $[-\pi, \pi]$, then Theorem 2 is easily obtained from previously known theorems.

** The definition of almost uniformly convergent sequences is given in (1), p. 27. The existence of a sequence of functions $h_m(x)$ with the stated properties follows from Lemma 3.3 in (1), p. 27.

where $[a]$ denotes the integer part of the number a, and

$$\Omega_m = \max_{x \in [-\pi,\pi]} |w_m(x) - w_{m-1}(x)| \qquad (m = 1, 2, ...),$$

$$r_0 = 0, \qquad r_m = \sum_{s=1}^{m} \mu_s \qquad (m = 1, 2, ...),$$

$$Q_0(x) = 0, \qquad Q_t(x) = w_{m-1}(x) + (t - r_{m-1}) \frac{w_m(x) - w_{m-1}(x)}{\mu_m}$$

$$(r_{m-1} < t \leq r_m, \ m = 1, 2, ...).$$

It is easy to see that the functions $Q_t(x)$ are continuous on $[-\pi, \pi]$ and are defined for all $t = 0, 1, 2, \ldots$. We now define an increasing sequence of natural numbers M_t $(t = 0, 1, 2, ...)$ and the trigonometric series (6) in the following way.

Put $M_0 = 1, \ a_0 = a_1 = b_1 = 0$. Suppose next that the natural number M_{t-1} and the numbers a_j, b_j for $j = 1, 2, \ldots, M_{t-1}$, where t is some natural number, have already been defined. Put

$$S_{t-1,0}(x) = 0, \qquad S_{t-1,k}(x) = \begin{cases} \sum_{j=1}^{k} (a_j \cos jx + b_j \sin jx), & (1 \leq k \leq M_{t-1}), \\ \sum_{j=1}^{M_{t-1}} (a_j \cos jx + b_j \sin jx), & (M_{t-1} < k), \end{cases}$$

$$D_{t-1,n}(x, T) = \sum_{k=0}^{\infty} a_{nk} S_{t-1,k}(x).$$

It is easy to see that, for a given t, the sequence of functions $D_{t-1,n}(x, T)$ $(n = 0, 1, 2, ...)$ converges uniformly on $[-\pi, \pi]$ to the function

$$S_{t-1,M_{t-1}}(x) = \sum_{j=1}^{M_{t-1}} (a_j \cos jx + b_j \sin jx),$$

as $n \to \infty$. In this case we can define a natural number M'_t satisfying the conditions

$$M'_t > M_{t-1}, \qquad \gamma_t < 1/2^t,$$

where

$$\gamma_t = \sup_{n,n' > M'_t} \max_{x \in [-\pi,\pi]} |D_{t-1,n}(x,T) - D_{t-1,n'}(x,T)|.$$

We now apply Lemma A, in which we put

$$\varphi(x) = Q_t(x) - D_{t-1,M_{t-1}}(x,T), \tag{9}$$

$$\sigma = 1/2^t, \qquad L = M_{t-1}, \qquad L_0 = M'_t. \tag{10}$$

Then, on the basis of this lemma, we can define a natural number $M_t > M_{t-1}$, a trigonometric polynomial

$$H_t(x) = \sum_{j=M_{t-1}+1}^{M_t} (a_j \cos jx + b_j \sin jx) \tag{11}$$

and the sets

$$E_t, \qquad G_{t,n} \qquad (M_{t-1} < n \leq M_t),$$

which satisfy all the conditions of Lemma A, if they are taken correspondingly-

respectively, instead of L', $H(x)$, E, and G_n, and define $\varphi(x)$, σ, L, and L_0 from equalities (9) and (10).

Thus we define trigonometric polynomials $H_t(x)$ for all $t = 1, 2, ...$, where the natural numbers M_t $(t = 0, 1, 2, ...)$ form an increasing sequence. Since we have set $M_0 = 1$, $a_0 = a_1 = b_1 = 0$, all terms of the trigonometric series (6) will be determined. It can be shown that the trigonometric series thus obtained satisfies all the conditions of Theorem 1.

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Note: Figure translations are in progress. See original paper for figures.