MATHEMATICS

1967

SovietRxiv

Abstract

Full Text

MATHEMATICS

I. MAREK (I. MAREK)

A GENERALIZED MINIMAX PRINCIPLE FOR INDECOMPOSABLE OPERATORS

(Presented by Academician S. L. Sobolev on 22 XII 1966)

The main purpose of this note is to generalize the minimax principle known for indecomposable matrices with nonnegative elements ([1],[2]).

Let Y be a real Banach space, X the corresponding complex extension; Y', X' the conjugate Banach spaces of continuous linear functionals on Y, X, and $[X]$, $[Y]$ the Banach spaces of bounded linear mappings of the spaces X, Y into themselves. It is assumed that $K \subset Y$ is a reproducing and normal cone ([3]). By means of the cone K in Y a partial order is introduced: $x < y$ $(y > x) \iff y - x \in K$. The symbol K' denotes the cone conjugate to K, i.e. the set $\{x' \in Y' : \langle x, x'\rangle \geq 0,\ x \in K\}$, where $\langle x, x'\rangle$ is the value of the functional $x' \in Y'$ at the element $x \in Y$.

An operator $T \in [Y]$ is called **positive** ([3]) if $Tx \in K$ for $x \in K$. A positive operator $T \in [Y]$ is called **semi-nonsupporting** ([4]) if, for an arbitrary pair $x \in K,\ x \neq 0,\ x' \in K',\ x' \neq \vartheta$, where $0, \vartheta$ are the zero elements in Y and Y', respectively, there exists a positive integer $p = p(x, x')$ such that $\langle T^p x, x'\rangle > 0$. An element $x \in K$ is called a **nonsupporting element of the cone** K if $\langle x, x'\rangle > 0$ for all nonzero functionals $x' \in K'$ ([4]). A positive operator $T \in [Y]$ is called **strictly nonsupporting** ([4]) if, for an arbitrary element $x \in K,\ x \neq 0$, one can find a positive integer $p = p(x)$ such that $T^n x$ is a nonsupporting element of the cone K for $n \geq p$.

A set $H' \subset K'$ will be called K-**total** if from the conditions $\langle x, x'\rangle \geq 0$ for all $x' \in H'$ it follows that $x \in K$. We shall agree to say that an operator $T \in [X]$ has property (S) if every point $\lambda \in \sigma(T)$, $|\lambda| = \rho(T)$, where $\sigma(T)$ is the spectrum of the operator T and $\rho(T)$ is the spectral radius of the mapping T, is an isolated pole of the resolvent $R(\lambda, T) = (\lambda I - T)^{-1}$ (I is the identity operator). If $T \in [Y]$, then $\widetilde{T}$ denotes the complex extension of the operator T, i.e. the operator for which $\widetilde{T}z = Tx + iTy$, where $z = x + iy,\ x \in Y,\ y \in Y$. In this case, by definition, $\sigma(T) = \sigma(\widetilde{T})$, $\rho(T) = \rho(\widetilde{T})$.

Let T be a linear operator mapping a dense domain of definition $D(T) \subset X$ into X. The symbol T' denotes the operator conjugate to the operator T, i.e. the operator whose domain of definition is $D(T') = \{v' \in X' : T'v' \in X'\}$ and which is defined by the relations

$$T'v' = u' \iff \langle x, u' \rangle = \langle Tx, v' \rangle \quad (^5).$$

Let H' be a K-total set, T a semi-nonsupporting operator, and $x \in K,\ x \neq 0,\ x' \in K',\ x' \neq \vartheta$. Introduce the functionals $r_x, r^x, s_{x'}, s^{x'}$, setting

$$r_x = \inf_{\substack{x' \in H' \\ \langle x, x' \rangle \neq 0}} \frac{\langle Tx, x' \rangle}{\langle x, x' \rangle}, \qquad r^x = \sup_{x' \in H'} \frac{\langle Tx, x' \rangle}{\langle x, x' \rangle},$$

$$s_{x'} = \inf_{\substack{x \in K \\ \langle x, x' \rangle \neq 0}} \frac{\langle x, T'x' \rangle}{\langle x, x' \rangle}, \qquad s^{x'} = \sup_{x \in K} \frac{\langle x, T'x' \rangle}{\langle x, x' \rangle}.$$

An element $x \in K$ will be called **extremal with respect to the mapping** T if at least one of the conditions $r_x = \rho(T)$, $r^x = \rho(T)$ holds.

In what follows the following conditions will be used:

(a) K is a reproducing and normal cone.

(b) H' is a K-total set.

(c) $T \in [Y]$ is a semi-nonsupporting operator.

(c′) $T \in [Y]$ is a strictly nonsupporting operator.

(d) The operator T has property (S).

Theorem 1 (generalized minimax principle). *Under conditions (a), (b), (c), and (d):*

1. *The equalities hold*

$$\mu_0 = \rho(T) = \min_{\substack{x \in K \\ x \neq 0}} r^x = \max_{x \in K} r_x.$$

2. *The number μ_0 is an eigenvalue of the operator T, and to this value there corresponds an eigenvector $x_0 \in K$ such that from the conditions $y = \nu T y$, $y \in K$, $y \neq 0$, where ν is some positive number, it follows that $y = c x_0$, where c is some constant. Moreover, the vector x_0 is a nonsupporting element of the cone K.*

If, in addition, condition (c′) is fulfilled, then for every element $z \in K$ extremal with respect to the mapping T there is a $c > 0$ such that $z = c x_0$.

Remark 1. Condition (c) is equivalent to condition

(e) The operator $T \in [Y]$ is indecomposable, i.e., an operator T such that from the relations $\alpha x_0 \succ T x_0$ for some $x_0 \in K$, $x_0 \neq 0$, $\alpha > 0$, it follows that x_0 is a nonsupporting element of the cone K (⁶).

Remark 2. Suppose that instead of condition (c) the condition

(e) The operator $T \in [Y]$ is u_0-positive ([7,11])

is satisfied. Instead of condition (c′), the condition

(e′) The operator $T \in [Y]$ is uniformly u_0-positive

is satisfied. Under these assumptions Theorem 1 was proved by the author in ([11]). Although in general the operator T' is not semi-nonsupporting ([8]), nevertheless the following theorem holds.

Theorem 2. *From conditions (a), (b), (c), and (d) it follows that the equalities*

$$\mu_0 = \rho(T) = \min_{\substack{x' \in K' \\ x' \neq \theta}} s^{x'} = \max_{x' \in K'} s_{x'}$$

are valid.

The validity of Theorems 1 and 2 follows from the following facts.

1. Let $x \in K$, $x \neq 0$, $x' \in K'$, $x' \neq \theta$,

$$y = \frac{1}{n} \sum_{k=1}^{n} \mu_0^{-k} T^k x, \qquad y' = \frac{1}{n} \sum_{k=1}^{n} \mu_0^{-k} {T'}^k x', \qquad n = 1, 2, \ldots$$

Then

$$r_x \leq r_y, \qquad r^y \leq r^x, \qquad s_{x'} \leq s_{y'}, \qquad s^{y'} \leq s^{x'}.$$

2. In the norm $[X]$ the equality ([9]) holds

$$\lim_{n \to \infty} \frac{1}{n} \sum_{k=1}^{n} \mu_0^{-k} T^k = B_1,$$

where B_1 is the unique term of the principal part of the Laurent expansion of the resolvent $R(\lambda, T)$ in a neighborhood of the pole μ_0 ([10])

$$R(\lambda, T) = \sum_{k=0}^{\infty} A_k (\lambda - \mu_0)^k + (\lambda - \mu_0)^{-1} B_1.$$

3. $r_z = \rho(T) = r^z$, $\quad s_{z'} = \rho(T) = s^{z'}$ for $z = B_1 x$,

$$x \in K, \quad x \neq 0, \quad z' = B_1' x', \quad x' \in K', \quad x' \neq \vartheta.$$

In conclusion we present one more form of the generalized minimax principle, in which it can be applied in solving certain eigenvalue problems for differential equations.

Let $D(L), D(C)$ be dense sets in the space Y, and let them be the domains of definition of mappings L and C of these domains into Y. Suppose that the inverse operator L^{-1} exists and belongs to $[Y]$. Put $T_1 = L^{-1}C$, if $C \in [Y]$, and $T_2 = CL^{-1}$, if C is an unbounded operator whose domain of definition $D(C)$ contains the range of the operator L^{-1}.

Theorem 3. Suppose that for $T = T_j$, $j = 1, 2$, the conditions of Theorem 1 are satisfied. Then the equalities

$$\lambda_0 = \max_{x \in D(L) \cap K} \inf_{\substack{y' \in N' \\ \langle Lx, y' \rangle \neq 0}} \frac{\langle Cx, y' \rangle}{\langle Lx, y' \rangle} = \min_{0 \neq x \in D(L) \cap K} \sup_{y' \in N'} \frac{\langle Cx, y' \rangle}{\langle Lx, y' \rangle}$$

hold for $j = 1$, and

$$\lambda_0 = \max_{y \in P} \inf_{\substack{x' \in D(L') \cap K' \\ \langle y, L'x' \rangle \neq 0}} \frac{\langle y, C'x' \rangle}{\langle y, L'x' \rangle} = \min_{0 \neq y \in P} \sup_{x' \in D(L') \cap K'} \frac{\langle y, C'x' \rangle}{\langle y, L'x' \rangle}$$

for $j = 2$, where λ_0 is the characteristic value of least modulus of the equation $Lx = \lambda Cx$, and we put $P = \{y = L^{-1}x : x \in K\}$, $N' = \{y' = (L^{-1})'x' : x' \in K'\}$.

Mathematical Institute
of Charles University
Prague, Czechoslovakia

Received
8 XII 1966

REFERENCES

1. F. R. Gantmakher, *Theory of Matrices*, Moscow, 1953.
2. H. Wielandt, Math. Zs., **52**, 642 (1950).
3. M. G. Krein, M. A. Rutman, UMN, **3**, No. 1 (1948).
4. I. Sawashima, Nat. Sci. Rep. Ochanomizu Univ., **15**, No. 1 (1964).
5. G. Köthe, Math. Ann., **159**, No. 5, 309 (1965).
6. V. Ya. Stetsenko, UMN, **21**, No. 5 (1966).
7. M. A. Krasnosel' skii, *Positive Solutions of Operator Equations*, Moscow, 1962.
8. I. Sawashima, Nat. Sci. Rep. Ochanomizu Univ., **17**, No. 1 (1966).

9. I. Marek, Mat. fyz. čas., **14**, No. 4 (1964).

10. A. E. Taylor, *Introduction to Functional Analysis*, N. Y., 1958.

11. I. Marek, SIAM J., **15**, No. 3 (1967).

Note: Figure translations are in progress. See original paper for figures.