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Abstract

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MATHEMATICS

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A GENERALIZED RIEMANN-LIOUVILLE OPERATOR AND SOME OF ITS APPLICATIONS

In the author' s monograph ([1]), the operator of fractional integro-differentiation of Riemann—Liouville found an essential application in the theory of the parametric representation of meromorphic functions.

In the present note we give the construction of a generalized operator of Riemann —Liouville type, which makes it possible, from a considerably more general point of view, to approach the solution of questions in the theory of classes of analytic functions and to establish a number of new results in this direction. Below we give the statements of these results, as well as a new sufficient condition for the solvability of the Hausdorff moment problem that is naturally connected with them.

1°. Denote by Ω the set of functions $\omega(x)$ satisfying the conditions:

1) $\omega(x)$ is nonnegative and continuous on $[0,1)$, and

$$\omega(0)=1, \qquad \int_0^1 \omega(x)\,dx<+\infty;$$

2) for every r $(0\le r<1)$

$$\int_r^1 \omega(x)\,dx>0.$$

Further, for $\omega(x)\in\Omega$ define the function

$$p(0)=1, \qquad p(\tau)=\tau\int_\tau^1 \frac{\omega(x)}{x^2}\,dx, \qquad \tau\in(0,1], \tag{1}$$

continuous on $[0,1]$, and we shall agree to write $p(\tau)\in P_\omega$.

Finally, for $p(\tau)\in P_\omega$ introduce into consideration the function

$$\Delta(r) = (1+r)\int_0^1 \tau^r \, dp(\tau), \qquad r \in [0, +\infty), \tag{2}$$

continuous on the half-axis $[0, +\infty)$ and representable also in the form

$$\Delta(0) = 1, \qquad \Delta(r) = r\int_0^1 \omega(x)x^{r-1}\, dx, \quad r \in (0, +\infty). \tag{2'}$$

Having then defined the sequence of positive numbers

$$\Delta_k = \Delta(k) \qquad (k = 0, 1, 2, ...),$$

we introduce into consideration the power series

$$C(z;\omega) = \sum_{k=0}^{\infty} \Delta_k^{-1} z^k, \qquad S(z;\omega) = 1 + 2\sum_{k=1}^{\infty} \Delta_k^{-1} z^k, \tag{3}$$

whose radius of convergence is equal to unity.

Concerning these functions in a neighborhood of their singular point $z = 1$, the following has been established.

Lemma 1. a) If

$$0 < \underline{\lim}_{x\to 1-0}\, \omega(x) \le \overline{\lim}_{x\to 1-0}\, \omega(x) < +\infty,$$

then

$$0 < \underline{\lim}_{r\to 1-0}(1-r)C(r;\omega) < +\infty.$$

b) If $\lim\limits_{x\to 1-0} \omega(x) = 0$, then $\lim\limits_{r\to 1-0}(1-r)C(r;\omega) = \infty$.

c) If $\lim\limits_{x\to 1-0} \omega(x) = +\infty$, then $\lim\limits_{r\to 1-0}(1-r)C(r;\omega) = 0$.

A more delicate property of the functions $C(z;\omega)$ and $S(z;\omega)$ is contained in the following theorem:

Theorem 1. If the function $\omega(x) \in \Omega$ is nondecreasing on $[0, 1)$, then

$$\operatorname{Re} C(z;\omega) \ge 0, \qquad \operatorname{Re} S(z;\omega) \ge 0 \qquad (|z| < 1). \tag{4}$$

Let us note that in the special case when $\omega(x) = (1-x)^\alpha$ $(-1 < \alpha < +\infty)$, we immediately obtain

$$\Delta_k = \Gamma(1+\alpha)\Gamma(1+k)/\Gamma(1+\alpha+k) \qquad (k=0,1,...), \tag{5}$$

$$C(z;\omega) = \frac{1}{(1-z)^{1+\alpha}}, \qquad S(z;\omega) = \frac{2}{(1-z)^{1+\alpha}} - 1.$$

In this case, for the values $-1 < \alpha \le 0$, property (4) of these functions is easily verified.

2°. Assuming that $\omega(x) \in \Omega$ and $p(\tau) \in P_\omega$, we introduce into consideration the operator

$$L^{(\omega)}[\varphi(x)] \equiv -\frac{d}{dx}\left\{x\int_0^1 \varphi(x\tau)\,dp(\tau)\right\}, \qquad x \in (0,1), \tag{6}$$

assuming that, on the appropriate classes of admissible functions, its right-hand side exists at least almost everywhere on $(0,1)$.

Let us note that in the simplest case, when $\omega(x) \equiv 1$, as is easy to see, almost everywhere

$$L^{(\omega)}[\varphi(x)] = \varphi(x)$$

for $\varphi(x) \in L(0,1)$. In the case $\omega(x) = (1-x)^\alpha$ $(-1 < \alpha < +\infty)$, one can establish the formula

$$L^{(\omega)}[\varphi(x)] = \Gamma(1+\alpha)x^{-\alpha}D^{-\alpha}\varphi(x), \qquad x \in (0,1), \tag{7}$$

where $D^{-\alpha}$ is the Riemann-Liouville operator, i.e.

$$D^0\varphi(x) = \varphi(x), \qquad D^{-\alpha}\varphi(x) = \frac{1}{\Gamma(\alpha)}\int_0^x (x-t)^{\alpha-1}\varphi(t)\,dt \qquad (0 < \alpha < +\infty),$$

$$D^{-\alpha}\varphi(x) = \frac{d}{dx}D^{-(1+\alpha)}\varphi(x) \qquad (-1 < \alpha < 0). \tag{8}$$

In the general case, under certain additional conditions imposed on the admissible functions $\omega(x) \in \Omega$ or on the functions $\varphi(x)$, the operator $L^{(\omega)}$ can be represented in other forms.

Lemma 2. a) Let $\omega(x) \in \Omega$ be continuous on $[0,1]$, $\omega(1) = 0$, and, in addition, $\omega'(x) \in L(0,1)$. Then, in the class of bounded summable functions $\varphi(x)$ on $(0,1)$,

$$L^{(\omega)}[\varphi(x)] = -\int_0^1 \varphi(x\tau)\omega'(\tau)\,d\tau. \tag{9}$$

b) Let $\omega(x) \in \Omega$. Then in the class $C_1[0,1]$ of functions $\varphi(x)$ continuously differentiable on $[0,1]$,

$$L^{(\omega)}[\varphi(x)] = \varphi(0) + x\int_0^1 \varphi'(x\tau)\omega(\tau)\,d\tau. \tag{10}$$

Finally, let us note the formulas

$$L^{(\omega)}[1] = 1, \qquad L^{(\omega)}[x^r] = \Delta(r)x^r, \qquad r \in (0,+\infty), \quad x \in [0,1], \tag{11}$$

which follow from (10) and (2′).

By a direct application of formulas (10) and (11) one establishes

Theorem 2. a) Let the function

$$f(re^{i\varphi}) = \sum_{k=0}^{\infty} a_k (re^{i\varphi})^k \tag{12}$$

holomorphic in the disk $|z| < R$. Then the function

$$L^{(\omega)}[f(re^{i\varphi})] \equiv f_{(\omega)}(re^{i\varphi}) = \sum_{k=0}^{\infty} \Delta_k a_k (re^{i\varphi})^k \tag{13}$$

is holomorphic in the same disk $|z| < R$.

b) For any ρ $(0 < \rho < R)$ the integral formulas

$$f(z) = \frac{1}{2\pi}\int_0^{2\pi} C\left(e^{-i\theta}\frac{z}{\rho};\omega\right) f_{(\omega)}(\rho e^{i\theta})\,d\theta \qquad (|z| < \rho), \tag{14}$$

$$f(z) = i\,\mathrm{Im}\,f(0) + \frac{1}{2\pi}\int_0^{2\pi} S\left(e^{-i\theta}\frac{z}{\rho};\omega\right) \mathrm{Re}\,f_{(\omega)}(\rho e^{i\theta})\,d\theta \qquad (|z| < \rho). \tag{15}$$

3°. We shall now give a representation of some general classes of harmonic and analytic functions in the disk $|z| < 1$ associated with the given function $\omega(x) \in \Omega$. We note that in the case $\omega(x) \equiv 1$ these classes and their representations are well known (see, for example, (2)). The more general case $\omega(x) = (1-x)^\alpha$ $(-1 < \alpha < +\infty)$ was recently considered by us (1).

Denote by U_ω the set of functions $u(z)$ harmonic in the disk $|z| < 1$ for which

$$U[u;\omega] = \sup_{0 \le r < 1} \left\{ \int_0^{2\pi} |u_{(\omega)}(re^{i\varphi})|\,d\varphi \right\} < +\infty, \tag{16}$$

where, as usual, $\omega(x) \in \Omega$ and $u_{(\omega)}(re^{i\varphi}) = L^{(\omega)}[u(re^{i\varphi})]$. The function

$$P(\theta, r; \omega) = \operatorname{Re} S(re^{i\theta}; \omega) = 1 + 2\sum_{k=1}^{\infty} \Delta_k^{-1} r^k \cos k\varphi \tag{17}$$

belongs to the class U_ω, since, by virtue of (10) and (11),

$$L^{(\omega)}[P(\theta, r; \omega)] = 1 + 2\sum_{k=1}^{\infty} r^k \cos k\varphi = \frac{1 - r^2}{1 - 2r\cos\theta + r^2} \equiv P(\theta, r) \geq 0, \tag{18}$$

and thus $P_{(\omega)}(\theta, r; \omega)$ is the Poisson kernel. We also note that if $\omega(x) \in \Omega$ does not decrease on $[0, 1)$, then, according to Theorem 1, we shall have

$$P(\theta, r; \omega) \geq 0 \qquad (0 \leq r < 1,\ 0 \leq \theta \leq 2\pi). \tag{19}$$

Theorem 3. a) The class U_ω coincides with the set of functions of the form

$$u(re^{i\varphi}) = \frac{1}{2\pi}\int_0^{2\pi} P(\varphi - \theta, r; \omega)\, d\psi(\theta), \tag{20}$$

where $\psi(\theta)$ is an arbitrary function of bounded variation on $[0, 2\pi]$, and, for some sequence of numbers $\rho_n \uparrow 1$,

$$\psi(\theta) = \lim_{n\to\infty} \int_0^{\theta} u_{(\omega)}(\rho_n e^{i\varphi})\, d\varphi, \qquad \theta \in [0, 2\pi]. \tag{21}$$

b) The class $U_\omega^* \subset U_\omega$ of functions $u(z)$ for which $u_{(\omega)}(z) \geq 0$ $(|z| < 1)$ coincides with the set of functions of the form (20), where $\psi(\theta)$ is an arbitrary nondecreasing function.

In the case $\omega(x) \equiv 1$, this theorem contains the known theorem (2) on the class of functions U representable by the Poisson–Stieltjes integral, since then $u_{(\omega)}(z) \equiv u(z)$ and $P(\theta, r, 1) \equiv P(\theta, r)$.

Relying on Lemma 1 and Theorem 3, one can prove:

a) $U_\omega \subset U$, if $\omega(x) \uparrow \infty$ as $x \uparrow 1$;

b) $U \subset U_\omega$, if $\omega(x) \downarrow 0$ as $x \uparrow 1$,

and both inclusions are proper.

From Theorem 3 there follows the following generalization of the Herglotz theorem (3).

Theorem 4. The class C_ω of functions $f(z)$ analytic in the disk $|z| < 1$ for which

$$\operatorname{Re} f_{(\omega)}(z) \geq 0 \qquad (|z| < 1),$$

coincides with the set of functions of the form

$$f(z) = iC + \frac{1}{2\pi} \int_0^{2\pi} S(e^{-i\theta} z; \omega)\, d\psi(\theta) \quad (|z| < 1), \tag{23}$$

where $\psi(\theta)$ is an arbitrary function of bounded variation on $[0, 2\pi]$.

4°. We now give an application of the functions of the class Ω to the Hausdorff moment problem ([4,5]), which also allows us to construct the inverse of the operator $L^{(\omega)}$ in the case when $\omega(x)$ is a nondecreasing function.

Theorem 5. Let the function $\omega(x) \in \Omega$ on $[0, 1]$ be nondecreasing. Then:

a) There exists a nondecreasing function $a_\omega(x)$, bounded on $[0, 1]$, continuous at the point $x = 0$, and such that the function

$$U(r) = \Delta^{-1}(r) = \left\{ r \int_0^1 \omega(x) x^{r-1}\, dx \right\}^{-1} \tag{24}$$

admits the representation

$$U(r) = \int_0^1 x^r\, da_\omega(x), \qquad r \in [0, +\infty). \tag{25}$$

b) The Hausdorff moment problem

$$\mu_n = \Delta_n^{-1} = \int_0^1 x^n\, da(x) \qquad (n = 0, 1, 2, \ldots)$$

has the solution $a(x) = a_\omega(x)$ in the class of nondecreasing functions bounded on $[0, 1]$.

We note that for $\omega(x) \equiv 1$ we also have $U(r) = \Delta^{-1}(r) \equiv 1$, $r \in [0, +\infty)$, and therefore in this case, in the representation (25), the function $a_\omega(x)$ must have the form

$$a_\omega(x) \equiv c \quad (0 \leq x < 1), \qquad a_\omega(1) = c + 1. \tag{26}$$

Finally, under the assumption that the function $\omega(x) \in \Omega$ on $[0, 1]$ is nondecreasing, along with the operator $L^{(\omega)}[\varphi(x)]$ we consider the integral operator

$$M^{(\omega)}[\varphi(x)] \equiv \int_0^1 \varphi(x\tau)\, da_\omega(\tau), \qquad x \in (0,1), \tag{27}$$

assuming that $\varphi(x)$ is at least continuous on $[0,1]$, while $a_\omega(\tau)$ is the function whose existence was asserted in Theorem 5.

We note that in the case $\omega(x) \equiv 1$ we shall have $M^{(\omega)}[\varphi(x)] \equiv \varphi(x)$, since then $a_\omega(x)$ is determined by (26). In the more general case, when $\omega(x) = (1-x)^\alpha$ $(-1 < \alpha < 0)$, one can, along with (7), establish the formula

$$M^{(\omega)}[\varphi(x)] = \Gamma^{-1}(1+\alpha) D^\alpha \{x^\alpha \varphi(x)\}. \tag{28}$$

In this case the operators $L^{(\omega)}$ and $M^{(\omega)}$ will be inverses of each other, since for $-1 < \alpha < 0$ the operators D^α and $D^{-\alpha}$ are known ([1]) to possess this property. In the general case the following is true.

Theorem 6. In the class $C_1[0,1]$ of functions $\varphi(x)$ continuously differentiable on $[0,1]$, the operators $L^{(\omega)}$ and $M^{(\omega)}$ are inverses of each other; that is, for any function $\varphi(x) \in C_1[0,1]$,

$$L^{(\omega)} M^{(\omega)}[\varphi(x)] \equiv M^{(\omega)} L^{(\omega)}[\varphi(x)] \equiv \varphi(x), \qquad x \in [0,1]. \tag{29}$$

In conclusion, we note that the operators $L^{(\omega)}$ and $M^{(\omega)}$, as well as their special cases when $\omega(x) = (1-x)^\alpha$ $(-1 < \alpha < +\infty)$, can also be applied in the theory of the representation of meromorphic functions.

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