ON THE PROBLEM OF THE ENCOUNTER OF MOTIONS

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Abstract

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CYBERNETICS AND CONTROL THEORY

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ON THE PROBLEM OF THE ENCOUNTER OF MOTIONS

Consider the problem ([1–6]) of the minimax time to encounter of the pursuing ($y(t)$) and pursued ($z(t)$) controlled motions, described respectively by the equations

$$\dot{y} = Ay + Bu, \qquad \dot{z} = Az + Bv \tag{1}$$

under the condition that the resources of the controls $u(t)$ and $v(t)$, which may be used for $t \geq \tau$ for each current time instant τ, are constrained by the integral conditions

$$\int_{\tau}^{\infty} \|u(t)\|^2 \, dt \leq \mu^2(\tau), \qquad \int_{\tau}^{\infty} \|v(t)\|^2 \, dt \leq \nu^2(\tau). \tag{2}$$

Here y, z are n-vectors; u, v are r-vectors; A, B are constant matrices of the corresponding dimensions. The quantities $\mu(\tau)$ and $\nu(\tau)$ change with increasing τ in accordance with the actual expenditure of the control resources. This leads to the equations

$$\dot{\mu} = -\|u\|^2/2\mu, \qquad \dot{\nu} = -\|u\|^2/2\nu. \tag{3}$$

The control $u(\tau)$ at each time instant τ, according to the conditions of the problem, must be formed by the feedback principle on the basis of measuring the quantities $y(\tau), z(\tau), \mu(\tau)$, and $\nu(\tau)$, i.e.

$$u(\tau) = u[y(\tau), z(\tau), \mu(\tau), \nu(\tau)]. \tag{4}$$

At the same time, information about the future choice of $v(t)$ $(t \geq \tau)$ is absent. For $v(\tau)$, both program control and control by the feedback principle are allowed, i.e. in the form

$$v(\tau) = v[y(\tau), z(\tau), \mu(\tau), \nu(\tau)]. \tag{5}$$

If the question is that of encounter in all phase coordinates $\{y_i\}$ and $\{z_i\}$, i.e., if it is required at the encounter instant $t = \tau + T$ to achieve the equality $y(\tau + T) = z(\tau + T)$, then the problem of $\min_u \max_v T$ is solved ([6]) by the rule of extremal aiming ([5–7]). This rule is as follows. From the measured $y(\tau), z(\tau), \mu(\tau), \nu(\tau)$, the attainability regions $G_1[\tau, T]$, $G_2[\tau, T]$ of the motions $y(t)$ and $z(t)$ are constructed, and the absorption instant $\tau + T^0(\tau)$ is found, when for the first time $G_2 \subset G_1$. Then $\min_u \max_v T = T^0$, and the optimal controls $u^0(\tau), v^0(\tau)$, formed by the feedback principle, are determined from the condition of program aiming of the motions $y(t)$ and $z(t)$ $(t \geq \tau)$ to the point ξ^*, where the boundaries of the regions G_1 and G_2 touch. We emphasize that, owing to the integral character of the constraints (2), the corresponding differential game does not, generally speaking, have a saddle point, since the extremal control v^0 does not ensure a time to encounter no less than T^0 (see Example 7.1 in the article ([6])).

The rule of extremal aiming is equivalent ([6]) to the rule: to the original problem there is assigned a problem of limiting speed,

$$y(\tau) - z(\tau) = x(\tau) \to x(\tau + T^0) = 0, \qquad T^0 = \min_w T \tag{6}$$

for the system

$$\dot{x} = Ax + Bw \tag{7}$$

under the condition

$$\xi(\tau) = \left[\int_\tau^\infty \|w(t)\|^2 \, dt \right]^{1/2} \leqslant \mu(\tau) - \nu(\tau); \tag{8}$$

then $T^0 = \min_u \max_v T$, and the optimal controls u^0, v^0 are determined by the equalities

$$u^0(\tau) = \frac{\mu}{\xi} \, w^0_\tau(\tau), \qquad v^0(\tau) = \frac{\nu}{\xi} \, w^0_\tau(\tau), \tag{9}$$

where $w^0_\tau(t)$ is the solution of problem (6), (7) under condition (8).

In the case where the meeting is considered only with respect to part of the coordinates $\{y_{i_j}\} = y_{[m]}$, $\{z_{i_j}\} = z_{[m]}$ $(j = 1, \dots, m < n)$, when at the moment of meeting it is required only that $y_{[m]}(\tau + T) = z_{[m]}(\tau + T)$, the situation becomes more complicated. The analysis of this situation constitutes the content of the present note.

The following assertions are valid:

1. *The choice of the control $u(\tau)$ from the condition of extremal aiming does not guarantee a meeting of the motions $y(t)$ and $z(t)$ in the time $\tau \leqslant t \leqslant \tau + T^0$, and this aiming does not ensure $\min_u \max_v T = T^0$.*

(The reachability regions G_1 and G_2 and the point ξ^* are now constructed, naturally, in the subspace $\{x_{i_j}\}$ $(j = 1, \dots, m)$.) This assertion is verified, for example, in the case of the motions

$$\dot{y}_1 = y_2, \qquad \dot{y}_2 = u_1, \qquad \dot{y}_3 = y_4, \qquad \dot{y}_4 = u_2, \tag{10}$$

$$\dot{z}_1 = z_2, \qquad \dot{z}_2 = v_1, \qquad \dot{z}_3 = z_4, \qquad \dot{z}_4 = v_2,$$

when it is required to bring about a meeting only in the coordinates y_1, y_3 and z_1, z_3. If in (10) $u = u^0(\tau)$ is determined by the rule of extremal aiming, and $v(\tau) = 0$, then for the initial data $y_1(0) = -0.25$, $y_2(0) = y_3(0) = y_4(0) = 0$, $\mu(0) = 1 + 2\sqrt{3}$, $z_1(0) = z_2(0) = z_3(0) = z_4(0) = 0$, $\nu(0) = 1$, even before the coincidence of $y_1(t), y_3(t)$ and $z_1(t), z_3(t)$, at the moment $\tau_* = 0.144$ the regions G_1 and G_2 merge, and at this moment the motion $z(t)$ obtains the possibility of slipping out of the reachability region $G_1(\tau, T)$ of the pursuing motion $y(t)$.

2. *It is impossible in general to construct a control $u(\tau) = u^*[y(\tau), z(\tau), \mu(\tau), \nu(\tau)]$ that would guarantee a meeting at a moment $t \leqslant \tau + T^0(\tau)$.* The assertion is verified by choosing the control $v^* = \dfrac{\nu}{\mu} u^*$.

3. Let $\varepsilon(\tau)$ and $T^0_\varepsilon(\tau)$ denote the quantities connected by the condition that $T^0_\varepsilon(\tau)$ is the solution of the problem

$$y(\tau) - z(\tau) = x(\tau) \to x_{[m]}(\tau + T^0_\varepsilon) = 0, \qquad T^0_\varepsilon = \min_w T_\varepsilon \tag{11}$$

for system (7) under the condition

$$\xi(\tau) = \left[\int_\tau^\infty \|w(t)\|^2 \, dt\right]^{1/2} \leqslant \mu(\tau) - \nu(\tau) - \varepsilon(\tau). \tag{12}$$

Then one can specify a control $u = u_0[y(\tau), z(\tau), \mu(\tau), \nu(\tau), \varepsilon(\tau)]$, ensuring a meeting no later than at the moment $t = \tau_0 + T^0_\varepsilon(\tau_0)$, whatever the initial data $y(\tau_0), z(\tau_0), \mu(\tau_0), \nu(\tau_0), \varepsilon(\tau_0) > 0$, for which problem (11), (7), (12) *has a solution.*

This assertion is verified by constructing the control u_0 in the form

$$u(\tau) = u_0[y(\tau), z(\tau), \mu(\tau), \nu(\tau), \varepsilon(\tau)] = w^0_\tau(\tau)\varphi[\zeta, \varepsilon]. \tag{13}$$

Here $w_\tau^0(t)$ is the solution of problem (11), (7) under condition (12). The number $T_\varepsilon^0(\tau)$ is computed in the course of the process as the least of the numbers T_ε^0 for which $\varepsilon(\tau) = \min[\varepsilon(\tau - 0), \varepsilon_0]$, where $\varepsilon_0 > 0$ is a number chosen in advance.

Finally, the function $\varphi[\zeta, \varepsilon]$ is chosen so as to ensure $\varepsilon(\tau) > 0$ for $t < \tau + T_\varepsilon^0(\tau)$. In particular, the functions

$$\varphi[\zeta, \varepsilon] = \left\{ \frac{\nu + \zeta}{\varepsilon\sqrt{\zeta}} \quad \text{for } 0 \leqslant \zeta < \varepsilon^2, \quad \frac{\nu + \zeta}{\zeta} \quad \text{for } \zeta \geqslant \varepsilon^2 \right\},$$

$$\varphi[\zeta, \varepsilon] = \left\{ \frac{\nu + \zeta}{\beta\varepsilon} \quad \text{for } 0 \leqslant \zeta < \beta\varepsilon, \quad \frac{\nu + \zeta}{\zeta} \quad \text{for } \zeta \geqslant \beta\varepsilon \right\},$$

$$\beta = \text{const} > 0.$$

Remark. The meaning of the control u_0 (13) is as follows: by means of a small additional reserve $\varepsilon(\tau) > 0$, a spring damping layer is, as it were, introduced between the boundaries of the regions G_1 and G_2, not allowing the region G_2 to merge with the region G_1 and, consequently, preventing the region G_2 from going beyond G_1 (see, in this connection, § 5 in paper ([8])). The change of $T_\varepsilon^0(\tau)$ in connection with the change of $\zeta(\tau)$ is relatively irregular in character (although the quantity $T_\varepsilon^0(\tau)$ does not increase monotonically). Therefore the control (13), generally speaking, does not close the system by differential feedback (in the sense in which this is defined in paper ([6])), and the above-indicated method of computing the quantity $T(\tau)$ should be strictly understood as the limiting case of changing the values of $\tau + T_\varepsilon^0(\tau)$ at discrete instants of time τ_k $(k = 0, 1, ...)$ on small intervals $[\tau_k, \tau_{k+1})$, between which the control (13) is determined with $\tau + T_\varepsilon^0(\tau) = \tau_k + T_\varepsilon^0(\tau_k)$ constant, while the quantity $\zeta(\tau)$ is then determined as the solution of the problem $y(\tau) - z(\tau) = x(\tau) \to x_{[m]}(\tau_k + T_\varepsilon^0(\tau_k)) = 0$ for system (7) under the condition

$$\zeta(\tau) = \left[\int_\tau^{\tau_k + T_\varepsilon{}^{0(\tau}{}_{k)}} \|w_\tau^0(t)\|^2 \, dt \right]^{1/2} = \min .$$

With the exception of the indicated circumstance, the definition of the class of admissible motions $y(t)$ and $z(t)$ does not essentially go beyond the apparatus of ordinary differential equations. Let us also note that only for exceptional, sufficiently unreasonable (from the standpoint of the interests of the motion $z(t)$) choices of the control $v(\tau)$ with $u = u_0(\tau)$ (13) is it possible for a kind of sliding regime to appear at $\zeta = 0$, which, however, leads to a rapid decrease of the quantity $T_\varepsilon^0(\tau)$.

Assertions (2) and (3) show that the control u_0 (13) ensures, in a certain sense, the maximum of what can be achieved from the standpoint of the motion $y(t)$ by means of an arbitrarily small increase in the control resource $u(\tau)$, regularizing the problem while preserving the condition of coincidence of the coordinates $\{y_{i_j}\}$ and $\{z_{i_j}\}$ at the moment of encounter. An analogous regularization is also achieved by bringing the point $\{y_{i_j}(t)\}$ at the moment of encounter into a small neighborhood of the point $\{z_{i_j}(t)\}$, but now without changing the initial reserve μ.

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Note: Figure translations are in progress. See original paper for figures.