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ONE CLASS OF SYMMETRIC SPACES WITH AN EXTENDABLE GROUP OF MOTIONS AND A GENERALIZATION OF THE POINCARÉ MODEL

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Let M be a symmetric space of affine connection, $\mathcal{G}(M)$ the group generated by displacements along geodesics. Generally speaking, the group $\mathcal{G}(M)$ is maximal in the sense that it is not contained in a broader (in dimension) Lie group of transformations of the space M, distinct from the group of motions. Therefore those cases in which such an extension is possible are of special interest. Below a construction will be indicated for one class of symmetric spaces with extendable group $\mathcal{G}(M)$.

1°. Let us recall some facts about transitive-differential groups of transformations ([1]). Let G be the Lie algebra of transformations of a (local) manifold M. Each operator from G is represented in the form of the series

$$a + A_1(x) + A_2(x, x) + \cdots, \tag{1}$$

where a is a vector of the tangent space T_0 to the manifold M at the initial point $(0, 0, \dots, 0)$, and $A_k(x, \dots, x)$ is a k-linear symmetric operator acting from $T_0 \times \cdots \times T_0$ into T_0. G is called the **Lie algebra of a transitive-differential group** of order ν (or simply the Lie algebra of a family D^ν) if:

1. together with the series (1), the algebra G also contains all the summands A_k.
2. $a \in G$, where a is an arbitrary vector from T_0.
3. G contains no operators of order higher than ν (i.e. $A_k = 0$ for $k > \nu$), but contains at least one operator $A_\nu \neq 0$.

We shall be interested only in the case $\nu = 2$. In this case every operator $A(x, x) \in G$ determines in the space T_0 a Jordan algebra J with multiplication law $xy = A(x, y)$, and, along with the operator $A(x, x)$, G contains all operators of the form

$$A(a; x, x) = 2(ax)x - ax^2 \qquad (a \in T_0).$$

Conversely, if J is an arbitrary Jordan algebra with multiplication law $A(x, y)$, then there exists at least one Lie algebra G of the family D^2 containing $A(x, x)$ as a quadratic operator; the manifold M on which the corresponding (local) group $\mathcal{G}$ acts may be considered to coincide with J. The minimal Lie algebra of this kind is the linear span of all possible vectors $a \in J$ (operators of order zero), the operators $A(a, x)$ and their commutators (linear operators), and, finally, the quadratic operators $A(x, x)$ and $A(a; x, x)$; we shall say that this Lie algebra is generated by the Jordan algebra J.

If the algebra J is semisimple, then the Lie algebra containing it of the family D^2 is unique.

2°. Let J be an arbitrary Jordan algebra (complex or real) with multiplication law $A(x, y)$, and let S be its linear transformation. Consider the Lie algebra G of the family D^2 generated by J. Operators of the form

$$a + A(S(a); x, x) \qquad (a \in J)$$

form a linear subspace $E_S \subset G$. Consider the subalgebra $H_S \subset G$, formed by all linear operators $A(x)$ for which $* \, [A(x), E_S] \subset E_S$. Obviously, $H_S = H_{-S}$.

Theorem. *If S is an involutive automorphism of the Jordan algebra J, then*

$$[E_\sigma, E_\sigma] \subset H_\sigma \qquad (\sigma = S \text{ or } -S),$$

and thereby the pair $(H_\sigma + E_\sigma, H_\sigma)$ determines on J the structure of a (local) symmetric space M_σ of an affine connection. If the algebra J contains an identity, then $G_\sigma = H_\sigma + E_\sigma$ is the Lie algebra of the group $\mathcal{G}(M_\sigma)$, generated by shifts along geodesics in the space M_σ.

It is obvious that G_σ is a proper subalgebra of the algebra G; consequently, the construction indicated above leads to (local) symmetric spaces M with extendable group $\mathcal{G}(M)$. Further, since the identity transformation of the algebra J is its involutive automorphism, at least one such space is associated with every Jordan algebra.

Let us note that the affine connection in M_σ is expressed at an arbitrary point x by the formula

$$\Gamma(x; u, v) = A_\sigma(f(x); u, v),$$

where u, v are two arbitrary vectors at the point x. Here

$$A_\sigma(a; u, v) = A(\sigma(a); u, v), \qquad f(x) = -2(x - x^3 + x^5 - \ldots),$$

where x^{2k+1} denotes

$$A_\sigma(\dots A_\sigma(A_\sigma(x;x,x);x,x)\dots;x,x),$$

i.e. the k-fold iteration of A_σ. Thus, at each point the connection is given by some Jordan algebra.

3°. With the theorem given above there is associated the following construction, which it is natural to regard as a generalization of the Poincaré model (conformal model) of Lobachevsky space to certain symmetric spaces of nonconstant curvature.

Let M be a homogeneous space with transformation group $\mathcal{G}$ and stationary subgroup $\mathcal{G}'$ (for the point $x_0 \in M$). Suppose that the Lie algebra G of the group $\mathcal{G}$ belongs to the family D^2. Starting from some quadratic operator $A(x,x)$ of the algebra G, and also from an involutive automorphism S (of the corresponding Jordan algebra), we construct, as above, the subspace E_S and the subalgebra $G_S = H_S + E_S$; denote by $\mathcal{G}_S$ the connected subgroup corresponding to this subalgebra. The orbit of the point x_0 with respect to $\mathcal{G}_S$ is some domain in M; denote it by M_S. Obviously,

$$M_S = \mathcal{G}_S/\mathcal{G}_S \cap \mathcal{G}'$$

is a locally symmetric space of an affine connection.

Analogously, the domain M_{-S} is a locally symmetric space of an affine connection, whose structure is dual to the structure of M_S.

We thus obtain a pair of mutually dual locally symmetric spaces M_S and M_{-S}, realized as domains in

* Square brackets denote the commutator in the Lie algebra.

the ambient space M. The motions of each of them extend to transformations belonging to some group $\mathcal{G}$, acting on M. The stationary subgroups of the initial point in M_S and M_{-S} have one and the same connected component of the identity (the latter corresponds to the subalgebra $H_S = H_{-S}$).

Especially interesting is the case when one of the groups $\mathcal{G}_S$ or $\mathcal{G}_{-S}$ is compact. Let, for example, $\mathcal{G}_S$ be compact. Then the domain M_S coincides with the whole space M and is a compact locally symmetric Riemannian space. The dual space M_{-S} is realized as a domain in M_S, and the motions of this domain are generated by transformations of the whole space M_S belonging to some extension of the group $\mathcal{G}(M_S)$.

In the case when the Jordan algebra J has an identity, the transformations from $\mathcal{G}$ have the invariant

$$(x_1, x_2, x_3, x_4) = \frac{\|x_1 - x_3\|}{\|x_2 - x_3\|} : \frac{\|x_1 - x_4\|}{\|x_2 - x_4\|},$$

where $\|x\|$ is the norm of the element $x \in J$. Therefore the distance $\rho_{-S}(x, y)$ in the space M_{-S}, up to a constant factor, is $|\ln(x, y, a, b)|$, where a, b are the infinitely distant points of the geodesic xy joining x and y in M_S (lying on the boundary of the domain M_{-S}).

The Poincaré model of n-dimensional Lobachevskii space may be regarded as a special case of this construction. As J one should take the n-dimensional simple Jordan algebra with multiplication law

$$xy = (e, x)y + (e, y)x - (x, y)e,$$

where x, y is a positive definite scalar product in the vector space J, and e is a fixed vector of length 1. Here $\|x\| = (x, x)$. The corresponding Lie algebra of the family D^2 is the Lie algebra of the conformal group acting on the n-dimensional sphere M (the space J, completed by the point ∞). Taking as S the operator of reflection in the plane $(e, x) = 0$, we obtain the Lie algebra of the rotation group of the sphere M (in stereographic coordinates $x^1, x^2, \dots, x^n$). The operator $-S$ leads to Lobachevskii space, realized as a hemisphere of the sphere in M.

As a second example let us consider the realization of the noncompact symmetric space $M_{-1} = SO(n, C)/SO(n)$ inside the dual (compact) space $M_1 = SO(n) \times SO(n)/SO(n)$ $(M_1 = SO(n))$. In this case J is the simple Jordan algebra of real skew-symmetric matrices of order n with multiplication law $xy = \frac{1}{2}(xky + ykx)$, where k is a fixed nondegenerate skew-symmetric matrix. Here $\|x\| = \det x$. The corresponding Lie algebra G of the family D^2 is the Lie algebra of matrices of the form

$$t = \begin{pmatrix} a & b \\ c & -a^T \end{pmatrix}$$

(a arbitrary, b, c skew-symmetric matrices of order n), acting in J as follows:

$$t(x) = b + ax + xa^T - xcx.$$

The group $\mathcal{G}$ of transformations corresponding to this Lie algebra is isomorphic to $SO(n, n)$. It consists of matrices of the form

$$\begin{pmatrix} p & q \\ r & s \end{pmatrix},$$

preserving the quadratic form $u_1 u_{n+1} + \dots + u_{nu_{2n}}$. The homogeneous space M is the set of pairs of matrices (y, z), defined up to up to simultaneous multiplication on the right by any nonsingular matrix and satisfying the condition $y^{Tz} + z^{Ty} = 0$. $\mathcal{G}$ acts in M as follows:

$$y' = py + qz, \qquad z' = ry + sz.$$

(If $\det z \neq 0$, then $(y, z) \sim (x, e)$, where e is the identity matrix and x is skew-symmetric. The algebra G was specified above by its action on the matrices x.)

The identity operator S corresponds to the space $M_1 = M$, and the operator $-S$ to the space M_{-1}, realized in M_1 as the domain

$$\{(x, e) \mid x^T = -x,\ xx^T < e\}.$$

The boundary of this domain is determined by the condition $\det(xx^T - e) = 0$.

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CITED LITERATURE

1. I. L. Kantor, DAN, 158, No. 6 (1964).

Note: Figure translations are in progress. See original paper for figures.