Conditions for the existence of recurrent trajectories in dynamic systems with a cylindrical phase space

Authors: E. A. Barbashin

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Abstract

The behavior of solutions to differential equations containing angular coordinates is investigated. Dynamical systems defined by such equations are often referred to as phase systems. In the first part of the work, a third-order equation is studied. Using a specially constructed Lyapunov function, an estimate of the domain of attraction of the zero equilibrium position is provided, provided that this position is asymptotically stable. The absence of stable equilibrium states entails the existence of recurrent motions distinct from equilibrium positions. Some of the obtained results are extended to higher-order equations.

Bibliography 5. Illustrations 1.

Full Text

Introduction

In 1967, E. A. Barbashin investigated the stability of solutions for third-order differential equations in [1]. We consider the equation:

$$\dddot{x} + a\ddot{x} + b\dot{x} + f(x) = 0 \tag{1}$$

where a and b are constants, and $f(x)$ is a continuous function such that $f(0) = 0$. We assume $f(x)$ satisfies the condition $\int_0^{2\pi} f(x)dx = 0$. Let x_1 and x_2 be roots of $f(x)$ in the interval $[0, 2\pi)$ such that $f'(x_1) < 0$ and $f'(x_2) < 0$.

By introducing the variables $y = \dot{x}$ and $z = \ddot{x}$, equation (1) can be rewritten as a system of first-order equations:

$$\begin{aligned} \dot{x} &= y \\ \dot{y} &= z \\ \dot{z} &= -az - by - f(x) \end{aligned} \tag{3}$$

To analyze the stability of the equilibrium point, we define a Lyapunov function $V(x, y, z)$. Under the assumption that $ab - f'(x) > 0$, the derivative of the Lyapunov function along the trajectories of the system is given by:

$$\dot{V} = (f'(x) - ab)y^2 \tag{4}$$

where $\dot{V} \leq 0$ when $y = 0$.

Stability Analysis

We examine the behavior of the system in the domain D defined by the coordinates (x, y, z). For the equilibrium point $(0, 0, 0)$ to be stable, specific conditions on the coefficients a and b must be met, namely $a > 0$ and $b > 0$. According to the Barbashin-Krasovskii theorem [4], if there exists a positive definite function V whose derivative $\dot{V}$ is negative semi-definite, and if the set where $\dot{V} = 0$ contains no trajectories other than the equilibrium point, then the equilibrium is asymptotically stable.

Consider the function:

$$V(x, y, z) = \int_0^x f(s)ds + \frac{1}{2}(ay + z)^2 + by^2 + \frac{1}{2}z^2 \tag{6}$$

The derivative of this function with respect to time, by virtue of system (3), is:

$$\dot{V} = -2aby^2 - 2az^2 - yf(x) \tag{8}$$

If $a > 0$ and $b > 0$, and the function $f(x)$ satisfies the conditions $f(x)x > 0$ for $x \neq 0$, the system exhibits global asymptotic stability. In cases where $f(x)$ is periodic or has multiple roots, such as x_1 and x_2, the analysis must be restricted to the neighborhood of these points.

Generalization to Higher-Order Systems

The results obtained for the third-order equation can be extended to n-th order differential equations of the form:

$$x^{(n)} + a_1 x^{(n-1)} + \cdots + a_{n-1}\dot{x} + f(x) = 0 \tag{11}$$

Using a similar transformation, we can represent this as a system of n first-order equations:

$$\begin{aligned} \dot{y}_1 &= y_2 \\ \dot{y}_2 &= y_3 \\ &\cdots \\ \dot{y}_n &= -a_1 y_n - a_2 y_{n-1} - \cdots - f(y_1) \end{aligned} \tag{15}$$

For these systems, the construction of a Lyapunov function follows the method of quadratic forms combined with integral terms of the nonlinearity $f(x)$. As shown in [3] and [5], the stability of the equilibrium point $(0, \ldots, 0)$ depends on the Hurwitz criteria for the linear part and the slope of the nonlinear function $f(x)$ at the origin. Specifically, if $f'(0) > 0$ and the coefficients a_i satisfy the Routh-Hurwitz conditions, the system is locally asymptotically stable.

Conclusion

The qualitative analysis of system (3) and its higher-order generalizations (15) demonstrates that the behavior of the trajectories is heavily influenced by the roots of the nonlinear function $f(x)$. When $f'(x_i) < 0$, the corresponding equilibrium point in the phase space (x, y, z) acts as a saddle point or an unstable node, whereas $f'(0) > 0$ typically corresponds to a region of attraction, provided the damping coefficients a and b are sufficiently large. These findings are consistent with the fundamental theorems of stability theory as presented by Barbashin and Nemytskii [1, 4].

References

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Note: Figure translations are in progress. See original paper for figures.