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Abstract

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MATHEMATICS

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ON THE RICHNESS OF THE CLASS OF QUASICONFORMAL MAPPINGS OF DOMAINS IN THREE-DIMENSIONAL EUCLIDEAN SPACE

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In the case of plane domains, every homeomorphic image of the unit disk whose boundary consists of more than one point is mapped quasiconformally (with arbitrary measurable and bounded distribution of characteristics) onto the disk. In 3-space, however, there exist homeomorphic images of the unit ball $|x| < 1$ which are not mapped quasiconformally onto the ball, but are bounded domains. The question arises: how powerful a subset do the quasiconformal images of the unit ball form in the set of all homeomorphic images of the latter. Below we set forth one of the approaches to solving this problem.*

In the space $\overline{H}$ of all bounded homeomorphic images of the unit ball, situated in the 3-space E_3 of Euclid, introduce the metric

$$\rho(D_1, D_2) = \max\left\{\sup_{x\in cD_1} \rho_{E_3}(x, cD_2),\ \sup_{x\in cD_2} \rho_{E_3}(x, cD_1)\right\}.$$

Here D_1 and D_2 are domains from the space $\overline{H}$; $cD = E_3 - D$; $\rho_{E_3}(A_1, A_2)$ is the ordinary distance between the sets A_1 and A_2 in E_3; x is a point of E_3. Let us note some properties of the metric ρ.

1. Relation with the convergence of sequences of domains to a kernel in the sense of Carathéodory.

Lemma 1. Let $\{D_n\}_{n=0}^{\infty}$ be a sequence of elements of $\overline{H}$. In order that the following conditions hold: 1) for every point x of D_0 there exists a section $\{D_n\}_{n=N}^{\infty}$ of the sequence $\{D_n\}_{n=0}^{\infty}$, for which D_0 is the kernel with respect to the point x; 2) there does not exist a subsequence of the sequence $\{D_n\}_{n=0}^{\infty}$ possessing a nondegenerate-to-a-point kernel with respect to some point x of cD_0, it is necessary and sufficient that there exist a ball R' intersecting all the domains $D_0, D_1, ...$, and that for every such ball R

$$\rho(R \cdot D_n,\ R \cdot D_0) \xrightarrow[n\to\infty]{} 0$$

(where $\rho(R \cdot D_n, R \cdot D_0)$ is defined in the same way as for elements of $\overline{H}$).

Corollary 1. Let $\{D_n\}_{n=0}^{\infty}$ be a sequence of elements of $\overline{H}$. If

$$\rho(D_n, D_0) \xrightarrow[n\to\infty]{} 0,$$

then for every point x of D_0 there exists a section $\{D_n\}_{n=N(x)}^{\infty}$ of the sequence $\{D_n\}_{n=0}^{\infty}$, for which D_0 is the kernel with respect to the point x and which converges to its kernel.

Corollary 2. If $\{D_n\}_{n=0}^{\infty} \subset \overline{H}$, $D_n \nearrow D_0$, then

$$\rho(D_n, D_0) \xrightarrow[n\to\infty]{} 0.$$

2. Relation with the property of lower semicontinuity of the Hersch coefficient of quasiconformality of a domain (^{1,2}). From (2) and Corollary 1 there follows the assertion:

* We consider only bounded homeomorphic images of the ball.

Lemma 2. If $\{D_n\}_{n=0}^{\infty} \subset H$ and $\rho(D_n, D_0) \xrightarrow[n\to\infty]{} 0$, then

$$K(D_0) \leq \varliminf_{n\to\infty} K(D_n).$$

Theorem. The subset Q of the metric space (H, ρ), consisting of domains admitting a quasiconformal mapping onto a ball, is an everywhere dense F_σ-set of the first category in this space.

Proof. Category. It suffices to prove that, for every number $C \geq 1$, the set Q_C of domains D in H such that $K(D) \leq C$ is nowhere dense in (H, ρ).

Fix a number C, $C \geq 1$, and consider an arbitrary ball

$$V = V(D_0, \varepsilon) = \{D : \ D \in H,\ \rho(D, D_0) < \varepsilon\}$$

in the space (H, ρ). Let x_0 be some point of the domain D_0, and let $|x - x_0| < r$ be a Euclidean ball belonging to the domain D_0 and having center x_0 and the greatest possible radius.

Next let a be one of the boundary points ∂D_0 of the domain D_0, lying on the sphere $|x - x_0| = r$. Make a radial slit in the ball $|x - x_0| < r$, and together

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with it in the domain D_0, of length $\min\{r, \varepsilon/2\}$. Denote the resulting domain by D_0^ε. D_0^ε is a homeomorphic image of a ball and has infinite coefficient of quasiconformality. The second assertion follows from the results of Väisälä (^{3}). It is also clear that $\rho(D_0, D_0^\varepsilon) = \min\{r, \varepsilon/2\}$. We assert that there exists a ball in the space (H, ρ) with center D_0^ε containing no domains from the set Q_C. Assuming the contrary, one could obtain a sequence $\{D_n\}$ of domains from Q_C such that $\rho(D_n, D_0^\varepsilon) \xrightarrow[n\to\infty]{} 0$. But then from Lemma 2 there follows the relation

$$K(D_0^\varepsilon) \leq \lim_{n\to\infty} K(D_n) \leq C,$$

which contradicts the infinitude of $K(D_0^\varepsilon)$. Therefore there exists a ball $V(D_0^\varepsilon, \delta)$ in the space (H, ρ), containing no elements of the set Q_C, whose radius δ is less than the number $\varepsilon/2$. The latter means that $V(D_0^\varepsilon, \delta) \subset V(D_0, \varepsilon)$ and, consequently, Q_C is nowhere dense in (H, ρ).

Density. Let D be an arbitrary domain in H and let ε be an arbitrary positive number. Our task is to find, in the ball $V(D, \varepsilon)$, a domain admitting a quasiconformal mapping onto a ball.

Let g be a homeomorphism of the ball $B = \{x : |x| < 1\}$ onto the domain D. There exists such a piecewise-linear homeomorphism h of the ball B in E_3 that $\rho_{E_3}[g(x), h(x)] < \varepsilon/3$ for all x in B, and any closed subdomain of the ball is represented as the sum of a finite number of pairwise nonintersecting sets, on each of which h is a linear transformation (^{4,5}).

Since $g(B_n) \nearrow g(B) = D$, $B_n = \{x : |x| < 1 - 1/n\}$, it follows, by virtue of Corollary 2 of Lemma 1, that there exists a natural number n_0 such that $\rho[g(B_{n_0}), D] < \varepsilon/3$. One can also calculate, using the inequality $\rho_{E_3}[g(x), h(x)] < \varepsilon/3$, $x \in B$, that $\rho[h(B_{n_0}), g(B_{n_0})] \leq 2\varepsilon/3$. Therefore $\rho[h(B_{n_0}), D] < \varepsilon$, and by virtue of the properties of the mapping h, the domain $h(B_{n_0})$ is a quasiconformal image of a ball.

It follows from Lemma 2 that the sets Q_C, $C \geq 1$, are closed in (H, ρ), and since

$$Q = \sum_{N=1}^{\infty} Q_N,$$

the theorem is completely proved.

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Note: Figure translations are in progress. See original paper for figures.

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