ON A PROBLEM OF STATIONARY HEAT CONDUCTION

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Abstract

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MATHEMATICAL PHYSICS

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ON A PROBLEM OF STATIONARY HEAT CONDUCTION

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In studying the kinetics of crystallization from a melt of semiconductor micromonocrystals on foreign substrates ([1,2]), grown by the method ([3,4]), it is necessary to determine the stationary temperature fields in the melt-substrate system. In the case where the crystallizing drops of melt are close in shape to a hemisphere, the following heat problem arises.

It is required to determine the temperature field $u(r,\theta)$ in the hemisphere $r \leq r_0$, $0 \leq \theta \leq \pi/2$, made of a material with coefficients of thermal conductivity K and external heat transfer h, lying along the surface of the great circle $\theta = \pi/2$ in perfect thermal contact with an infinite plane-parallel plate $-l \leq z \leq 0$ made of a material with coefficients K_1 and h_1, whose temperature is $u_1(\rho, z)$. Here r, θ denote spherical coordinates, ρ, z cylindrical coordinates, with the polar angle θ measured from the positive direction of the z-axis; $\theta = \pi/2$ corresponds to $z = 0$.

The functions $u(r,\theta)$ and $u_1(\rho, z)$ satisfy the following equations and boundary conditions:

$$\Delta_{r,\theta} u = 0, \qquad (\partial u/\partial r + hu)_{\substack{r=r_0 \\ 0\leq\theta\leq\pi/2}} = hu_0, \tag{1}$$

$$u\bigg|_{\substack{\theta=\pi/2 \\ r<r_0}} = u_1\bigg|_{\substack{z=0, \\ 0\leq\rho<r_0}}, \qquad -\frac{K}{r}\frac{\partial u}{\partial \theta}\bigg|_{\substack{\theta=\pi/2 \\ r<r_0}} = K_1 \frac{\partial u_1}{\partial z}\bigg|_{\substack{z=0 \\ 0\leq\rho<r_0}}, \tag{2}$$

$$\Delta_{\rho,z} u_1 = 0, \qquad (\partial u_1/\partial z + h_1 u_1)_{\substack{z=0 \\ \rho>r_0}} = h_1 u_0,$$

$$-K_1 \partial u_1/\partial z\bigg|_{z=-l} = q_0, \qquad \partial u_1/\partial \rho\bigg|_{\rho=\infty} = 0, \tag{3}$$

$$\Delta_{r,\theta} = \frac{\partial}{\partial r}\left(r^2 \frac{\partial}{\partial r}\right) + \frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta \frac{\partial}{\partial\theta}\right), \qquad \Delta_{\rho,z} = \frac{1}{\rho}\frac{\partial}{\partial\rho}\left(\rho\frac{\partial}{\partial\rho}\right) + \frac{\partial^2}{\partial z^2},$$

$$u_0 = \text{const}, \qquad q_0 = \text{const}.$$

The peculiarity of the problem is that the boundary conditions are prescribed on surfaces which are coordinate surfaces in different coordinate systems. We shall give approximate solutions for the cases $l \ll r_0$ and $l \gg r_0$; $h_1 r_0 \ll 1$.

1. For $l \ll r_0$, in order to determine the function $u(r,\theta)$, we introduce, proceeding from the expansion of $u_1(\rho, z)$ in a Maclaurin series in z, the approximate boundary condition

$$\left.\frac{1}{r}\frac{\partial u}{\partial\theta}\right|_{\theta=\pi/2} = \frac{q_0}{K}. \tag{4}$$

Then the solution of the problem (1), (4) can be expressed in terms of Legendre polynomials:

$$u(r,\theta) = u_0 - \frac{q_0 r_0}{K}\left[\frac{r}{r_0}\cos\theta + \nu\sum_{n=0}^{\infty} N'_n \left(\frac{r}{r_0}\right)^{2n} P_{2n}(\cos\theta)\right], \tag{5}$$

where

$$\nu = \frac{1+hr_0}{2}, \qquad N'_n = \frac{(4n+1)P_{2n}(0)}{(n+1)(2n-1)(2n+hr_0)}.$$

The result obtained can be used to determine the function $u(r,\theta)$ in the following approximation. Setting

$$u_1(\rho, z) = u_0 + \frac{q_0}{K_1}\frac{1-h_1 z}{h_1} + v(\rho, z), \tag{6}$$

by virtue of (2), (3), and (5), for the function $v(\rho, z)$ we shall have the boundary-value problem

$$\Delta_{\rho,z} v = 0, \qquad \left(\frac{\partial v}{\partial z} + h_1 v\right)_{z=0} = -\frac{q_0}{K_1}\Psi(\rho),$$

$$\left.\frac{\partial v}{\partial z}\right|_{z=-l} = 0, \qquad \left.\frac{\partial v}{\partial\rho}\right|_{\rho=\infty} = 0, \tag{7a}$$

where

$$\Psi(\rho) = \begin{cases} \alpha + \psi(\rho), & 0 \le \rho < r_0, \\ 0, & \rho > r_0, \end{cases} \qquad \psi(\rho) = \eta_1 \nu \frac{K_1}{K} \sum_{n=1}^{\infty} N_n \left(\frac{\rho}{r_0}\right)^{2n}, \tag{7b}$$

$$\alpha = 1 - \nu \frac{h_1}{h} \frac{K_1}{K}, \qquad \eta_1 = h_1 r_0, \qquad N_n = N'_{nP_{2n}}(0),$$

whence, instead of (4), there follows a boundary condition of the form

$$\frac{1}{r} \frac{\partial u}{\partial \theta}\bigg|_{\theta=\pi/2} = \frac{q_0}{k}[1 + X(r)], \tag{8}$$

$$X(\rho) = \Psi(\rho) + \frac{h_1 K_1}{q_0} \varphi(\rho), \qquad \varphi(\rho) = v(\rho, 0). \tag{9}$$

Applying the Hankel transforms to (7),

$$\bar{v} = \int_0^\infty \rho J_0(\rho\lambda) v \, d\rho, \qquad v = \int_0^\infty \lambda J_0(\rho\lambda) \bar{v} \, d\lambda, \tag{10}$$

whence

$$\varphi(\rho) = -\frac{q_0}{K_1} \int_0^\infty \frac{\lambda J_0(\rho\lambda) \operatorname{ch} \lambda l}{\lambda \operatorname{sh} \lambda l + h_1 \operatorname{ch} \lambda l} \int_0^{r_0} \rho' J_0(\rho'\lambda) \Psi(\rho') \, d\rho' \, d\lambda,$$

$$\Psi(\rho) = \int_0^\infty \lambda J_0(\rho\lambda) \int_0^{r_0} \rho' J_0(\rho'\lambda) \Psi(\rho') \, d\rho' \, d\lambda,$$

we obtain for the function $X(\rho)$ an expression through a Stieltjes integral

$$X(\rho) = \int_0^{r_0} \Psi(\rho') \frac{d}{d\rho'} [\rho' G(\rho, \rho')] \, d\rho', \tag{11}$$

where the function

$$G(\rho, \rho') = \int_0^\infty J_0(\rho\lambda) J_1(\rho'\lambda) \frac{\lambda \operatorname{th} \lambda l}{\lambda \operatorname{th} \lambda l + h_1} \, d\lambda \tag{12}$$

admits a representation in the form of the series

$$G(\rho,\rho') = 2h_1 \sum_{m=1}^{\infty} \frac{\beta_m I_0\left(\frac{\rho}{l}\beta_m\right) K_1\left(\frac{\rho'}{l}\beta_m\right)}{\beta_m^2 + h_1 l(1+h_1 l)}, \qquad \beta_m \operatorname{tg} \beta_m = h_1 l. \tag{13}$$

From the asymptotics of the cylindrical functions $I_0(x)$, $K_1(x)$ it follows that $G(\rho, r_0) \to 0$ as $l/r_0 \to 0$, $\rho < r_0$, $h_1 > 0$. On the other hand, from (11) we have

$$X(\rho) = \Psi(\xi) r_0 G(\rho, r_0), \qquad 0 \le \xi \le r_0, \tag{14}$$

moreover, according to (7b), the function $\psi(\rho)$ entering into Ψ is bounded,

$$|\psi(\rho)| \le \eta_1 \nu \frac{K_1}{K} N, \qquad N = \sum_{n=1}^{\infty} N_n < \ln 2, \tag{15}$$

and the parameter η_1 is in practice usually small. Since, moreover, the function $I_0(\rho\beta_1/l)$ changes slowly as ρ varies from 0 to r_0, the solution of the problem in this approximation will be well described by a formula of the form (5), with q_0/K replaced by $\dfrac{q_0}{K}(1+\overline{X})$, $\overline{X} = a r_0 G(\xi, r_0)$, $0 \le \xi \le r_0$.

2. For $r_0 \ll l$, the variation of the function $u(r,\theta)$ is small, which makes it possible, proceeding from the condition of balance of the heat flux through the hemisphere,

$$\iint\limits_{(S)} \frac{\partial u}{\partial r}\, dS = \iint\limits_{(S_0)} \frac{\partial u}{\partial z}\, dS_0$$

(S is the surface of the hemisphere $r = r_0$, S_0 is the circle $\theta = \pi/2$), to introduce, for determining the function $u_1(\rho, z)$ at $z = 0$, the effective boundary condition

$$\left(\frac{\partial u_1}{\partial z} + h_1' u_1\right)_{\substack{z=0\\ 0\le\rho<r_0}} = h' u_0, \qquad h' = \frac{K_1}{K}\frac{S}{S_0} h. \tag{16}$$

Then, by virtue of (3) and (16), for the function $v(\rho, z)$, defined by relation (6), the boundary-value problem will hold

$$\Delta_{\rho,z} v = 0, \qquad \left(\frac{\partial v}{\partial z} + h_1 v\right)_{z=0} = \mu \frac{q_0}{K_1} \Psi_1(\rho), \tag{17a}$$

$$\partial v/\partial z\big|_{z=-l} = 0, \qquad \partial v/\partial \rho\big|_{\rho=\infty} = 0,$$

where

$$\Psi_1(\rho) = \begin{cases} 1 + \dfrac{h_1 K_1}{q_0}\,\varphi(\rho), & 0 \le \rho < r_0, \\ 0, & \rho > r_0, \end{cases} \qquad \varphi(\rho) = v(\rho, 0), \qquad \mu = 1 - \frac{h'}{h_1}, \tag{17b}$$

and from (2), (6), and (17), in order to find the function $u(r,\theta)$ we obtain the boundary condition

$$\frac{1}{r}\frac{\partial u}{\partial \theta}\bigg|_{\theta=\pi/2} = \frac{h'}{h_1}\frac{q_0}{K}\Psi_1(r). \tag{18}$$

Applying the transform (10) to (17), for determining $\Psi_1(\rho)$ we shall have the integral equation

$$\Psi_1(\rho) = 1 + \mu h_1 \int_0^{r_1} \Psi_1(\rho')\,\frac{d}{d\rho'}\left[\rho' G_1(\rho,\rho')\right] d\rho', \tag{19}$$

where the kernel

$$G_1(\rho,\rho') = \int_0^\infty \frac{\operatorname{ch}\lambda l\, J_0(\rho\lambda) J_1(\rho'\lambda)}{\lambda \operatorname{sh}\lambda l + h_1 \operatorname{ch}\lambda l}\, d\lambda \tag{20}$$

can be expressed by the series

$$G_1(\rho,\rho') = 2\sum_{m=1}^{\infty} \frac{\beta_{mJ} 1\left(\dfrac{\rho'}{l}\beta_m\right) K_0\left(\dfrac{\rho}{l}\beta_m\right)}{\beta_m^2 + h_1 l(1 + h_1 l)}, \qquad \beta_m \operatorname{tg}\beta_m = h_1 l. \tag{21}$$

As a result of the smallness of r_0 ($\mu \sim 1$) in the present case, the solution of the boundary-value problem (1), (18) will practically have the form (5), with q_0/K replaced by

$$\frac{h' q_0}{h_1 K}.$$

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Note: Figure translations are in progress. See original paper for figures.