

**ON THE THEORY OF  
THE METHOD OF  
FINITE DIFFERENCES  
FOR SOLVING  
NONLINEAR  
BOUNDARY-VALUE  
PROBLEMS**

MATHEMATICS

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**Abstract**

**Full Text**

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*MATHEMATICS*

M. N. YAKOVLEV

**ON THE THEORY OF THE METHOD OF FINITE DIFFERENCES FOR SOLVING NON-LINEAR BOUNDARY-VALUE PROBLEMS**

*(Presented by Academician L. V. Kantorovich on 2 IV 1966)*

I. Consider the following Dirichlet problem. Find a function  $u(x, y)$ , twice continuously differentiable in the closed domain  $\bar{D}$ , satisfying the differential equation

$$F(x, y, u, u_x, u_y, u_{xx}, u_{yy}) = 0 \quad (1)$$

and assuming continuous values on the piecewise smooth boundary

$$u|_{\Gamma} = \varphi(x, y). \quad (2)$$

Here the function  $F(x, y, u, p, q, r, t)$  is continuously differentiable with respect to all its arguments and satisfies the inequalities

$$F_r(x, y, u, p, q, r, t) \geq 0, \quad F_t(x, y, u, p, q, r, t) \geq 0,$$

$$F_r(x, y, u, p, q, r, t) + F_t(x, y, u, p, q, r, t) \geq m > 0,$$

$$F_u(x, y, u, p, q, r, t) \leq 0$$

for  $x, y \in \bar{D}$ ,  $-\infty < u, p, q, r, t < +\infty$ .

Under these conditions, for the solution of the Dirichlet problem (1), (2), a uniqueness theorem is valid; the existence of the solution is assumed by us.

In the usual way we introduce a square mesh with step  $h$ , the mesh domain  $D_h$ , and its boundary in the sense of the five-point star  $\Gamma_h$ . We divide the boundary points into two groups  $\Gamma_h^{(1)}$  and  $\Gamma_h^{(2)}$ ,  $\Gamma_h = \Gamma_h^{(1)} + \Gamma_h^{(2)}$ . To  $\Gamma_h^{(1)}$  we assign all points of  $\Gamma_h$  that belong to  $\Gamma$ , and to  $\Gamma_h^{(2)}$  all the remaining points of  $\Gamma_h$ .

The boundary conditions are approximated in two ways:

A. **Simple transfer.** We set

$$\begin{aligned} u_{ij} &= \varphi(x_i, y_j), \quad \text{if } x_i, y_j \in \Gamma_h^{(1)}, \\ u_{ij} &= \varphi(\bar{x}_i, \bar{y}_j), \quad \text{if } x_i, y_j \in \Gamma_h^{(2)}. \end{aligned} \quad (3)$$

Here  $\bar{x}_i, \bar{y}_j$  is the point of  $\Gamma$  nearest to  $x_i, y_j$ .

B. **Collatz interpolation.** We set

$$\begin{aligned} u_{i,j} &= \varphi(x_i, y_j), \quad \text{if } x_i, y_j \in \Gamma_h^1, \\ u_{i,j} &= \frac{\delta}{\delta + h} u_{i,j-1} + \frac{h}{\delta + h} \varphi(\bar{x}_i, \bar{y}_j), \quad \text{if } x_i, y_j \in \Gamma_h^{(2)}. \end{aligned} \quad (4)$$

(For more details, see, for example, <sup>(1)</sup>.)

1. As the difference equation approximating the differential equation (1), consider the equation

$$\Gamma \left( x_i, y_j, u_{i,j}, \frac{u_{i+1,j} - u_{i-1,j}}{2h}, \frac{u_{i,j+1} - u_{i,j-1}}{2h}, \frac{u_{i+1,j} - 2u_{i,j} + u_{i-1,j}}{h^2}, \frac{u_{i,j+1} - 2u_{i,j} + u_{i,j-1}}{h^2} \right) = 0 \quad (5)$$

for  $x_i, y_j \in D_h$ .

**Theorem 1.** Let

$$|F_p(x, y, u, p, q, r, t)| \leq MF_r(x, y, u, p, q, r, t),$$

$$|F_q(x, y, u, p, q, r, t)| \leq MF_t(x, y, u, p, q, r, t)$$

for  $x, y \in \bar{D}$ ,  $-\infty < u, p, q, r, t < +\infty$ .

Then, for  $h < 1/M$ , the systems of difference equations (5), (3) and (5), (4) have solutions. These solutions are unique and

$$\lim_{h \rightarrow 0} \max_{x_i, y_j \in \bar{D}_h} |u(x_i, y_j) - u_{ij}| = 0.$$

If, in addition,  $u \in C^{(3)}(\bar{D})$ , then

$$\max_{x_i, y_j \in \bar{D}_h} |u(x_i, y_j) - u_{ij}| \leq Ch.$$

If  $u \in C^{(4)}(\bar{D})$  and Kolmat interpolation is used, then

$$\max_{x_i, y_j \in \bar{D}_h} |u(x_i, y_j) - u_{ij}| \leq C_1 h^2.$$

2. We now consider other ways of approximating the first derivatives  $u_x$  and  $u_y$ . Namely, if  $F_p(\dots) \geq 0$ , we replace  $u_x$  by  $(u_{i,j} - u_{i-1,j})/h$ ; if  $F_p(\dots) \leq 0$ , we replace  $u_x$  by  $(u_{i+1,j} - u_{i,j})/h$ . The treatment is analogous for  $u_y$ . In all, four cases occur. We state the result for one of them.

Consider the following difference equation approximating the differential equation (1):

$$F\left(x_i, y_j, u_{ij}, \frac{u_{i,j} - u_{i-1,j}}{h}, \frac{u_{i,j} - u_{i,j-1}}{h}, \frac{u_{i+1,j} - 2u_{i,j} + u_{i-1,j}}{h^2}, \frac{u_{i,j+1} - 2u_{i,j} + u_{i,j-1}}{h^2}\right) = 0. \quad (6)$$

**Theorem 2.** Let

$$F_p(x, y, u, p, q, r, t) \geq 0, \quad F_q(x, y, u, p, q, r, t) \geq 0,$$

$$F_u(x, y, u, p, q, r, t) \leq q_1 < 0$$

for  $x, y \in \bar{D}$ ,  $-\infty < u, p, q, r, t < +\infty$ .

Then the systems of difference equations (6), (4) and (6), (3) have solutions. These solutions are unique and

$$\lim_{h \rightarrow 0} \max_{x_i, y_j \in \bar{D}_h} |u(x_i, y_j) - u_{ij}| = 0.$$

If, moreover,  $u \in C^{(3)}(\bar{D})$ , then

$$\max_{x_i, y_j \in \bar{D}_h} |u(x_i, y_j) - u_{ij}| \leq C_3 h.$$

- II. We now consider a more general boundary-value problem. Find a function  $u \in C^{(2)}(\bar{D})$  satisfying equation (1) and the boundary condition

$$f(x, y, u, u_x, u_y)|_{\Gamma} = 0. \quad (7)$$

Here the function  $f(x, y, u, p, q)$  is continuously differentiable with respect to all its arguments and satisfies the inequalities

$$-f_u(x, y, u, p, q) \geq c > 0,$$

$$f_p(x, y, u, p, q) \cos(\mathbf{n}, x) \geq 0, \quad f_q(x, y, u, p, q) \cos(\mathbf{n}, y) \geq 0$$

for  $x, y \in \Gamma$ ,  $-\infty < u, p, q < \infty$ ;  $\mathbf{n}$  is the inward normal to the boundary  $\Gamma$ . We assume that the solution of problem (1), (7) exists and is unique.

We approximate the boundary conditions (7) as follows. Let the point  $x_i, y_j \in \Gamma_h$ . Draw the inward normal to the boundary  $\Gamma$  passing through the point  $x_i, y_j$ . We assume that the domain  $D$  is such that, for  $h < h_0$

( $h_0$  does not depend on  $x_i, y_j$ ) there exists a grid square with vertex at  $x_i, y_j$  such that the two vertices of this square adjacent to the vertex  $x_i, y_j$  belong to  $\bar{D}_h$ , and the normal constructed above passes inside or along the boundary of this square. For definiteness, suppose that the vertices indicated above have coordinates  $x_{i-1}, y_j$  and  $x_i, y_{j-1}$ . Denote by  $\bar{x}_i, \bar{y}_j$  the point of intersection of the normal drawn at  $x_i, y_j$  with the boundary  $\Gamma$ . Then we set

$$f\left(\bar{x}_i, \bar{y}_j, u_{i,j}, \frac{u_{i,j} - u_{i-1,j}}{h}, \frac{u_{i,j} - u_{i,j-1}}{h}\right) = 0 \quad (8)$$

for  $x_i, y_j \in \Gamma_h$ .

**Theorem 3.** Suppose

$$\begin{aligned} F_u(x, y, u, p, q, r, t) &\leq q_1 < 0, \\ |F_p(x, y, u, p, q, r, t)| &\leq MF_r(x, y, u, p, q, r, t), \\ |F_q(x, y, u, p, q, r, t)| &\leq MF_t(x, y, u, p, q, r, t) \end{aligned}$$

for  $x, y \in \bar{D}$ ,  $-\infty < u, p, q, r, t < +\infty$ .

Then, for sufficiently small  $h$ , the system of difference equations (5), (8) has a solution. This solution is unique and

$$\lim_{h \rightarrow 0} \max_{x_i, y_j \in \bar{D}_h} |u(x_i, y_j) - u_{i,j}| = 0.$$

If, moreover,  $u \in C^{(3)}(\bar{D})$ , then

$$\max_{x_i, y_j \in \bar{D}_h} |u(x_i, y_j) - u_{i,j}| \leq C_4 h.$$

If  $u \in C^{(4)}(\bar{D})$ ,  $f_p = f_q = 0$ ,  $\Gamma_h = \Gamma_h^{(1)}$ , then

$$\max_{x_i, y_j \in \bar{D}_h} |u(x_i, y_j) - u_{i,j}| \leq C_5 h^2.$$

In the case of sign-definiteness of the derivatives of the function  $F$  with respect to the variables  $u_x$  and  $u_y$ , and approximation of the differential equation in the form (6), theorems analogous to Theorem 2 hold. We state one of them.

**Theorem 4.** Under the assumptions of Theorem 2, the system of difference equations (6), (8) is uniquely solvable and

$$\lim_{h \rightarrow 0} \max_{x_i, y_j \in \bar{D}_h} |u(x_i, y_j) - u_{i,j}| = 0.$$

If, moreover,  $u \in C^{(3)}(\bar{D})$ , then

$$\max_{x_i, y_j \in \bar{D}_h} |u(x_i, y_j) - u_{i,j}| \leq C_6 h.$$

**Remark.** All results carry over to the case of any number of dimensions.

Leningrad Branch  
of the V. A. Steklov Mathematical Institute  
Academy of Sciences of the USSR

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1. L. Collatz, *Numerical Methods for Solving Differential Equations*, Moscow, 1953.

*Note: Figure translations are in progress. See original paper for figures.*

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