On a class of solutions of the sixth Painlevé equation

Authors: N. A. Lukashevich, A. I. Yablonskii

Date: 1967-01-01T00:00:00+00:00

Abstract

Full Text

Preamble

DIFFERENTIAL EQUATIONS 1967, VOLUME III, NO. 3

ON A CLASS OF SOLUTIONS OF THE SIXTH PAINLEVÉ EQUATION

N. A. LUKASHEVICH, A. I. YABLONSKII

Consider the sixth Painlevé equation:

$$\begin{aligned}\frac{d^2w}{dz^2} =&\frac{1}{2}\left(\frac{1}{w}+\frac{1}{w-1}+\frac{1}{w-z}\right)\left(\frac{dw}{dz}\right)^2-\left(\frac{1}{z}+\frac{1}{z-1}+\frac{1}{w-z}\right)\frac{dw}{dz}\\ &+\frac{w(w-1)(w-z)}{z^2(z-1)^2}\left[\alpha+\beta\frac{z}{w^2}+\gamma\frac{z-1}{(w-1)^2}+\delta\frac{z(z-1)}{(w-z)^2}\right]\end{aligned} \tag{1}$$

where $\alpha, \beta, \gamma, \delta$ are constants.

It is well known that the sixth Painlevé equation (1) defines a new transcendental function $w(z, \alpha, \beta, \gamma, \delta)$, which is a single-valued function of z except for the fixed singular points $z = 0, 1, \infty$. However, for certain specific values of the parameters $\alpha, \beta, \gamma, \delta$, equation (1) may possess solutions that can be expressed in terms of elementary or classical transcendental functions.

In this paper, we investigate a class of solutions for equation (1) that can be expressed through hypergeometric functions. We demonstrate that under certain constraints on the parameters, the sixth Painlevé equation can be reduced to a simpler form or related to the solutions of linear differential equations.

Specifically, we consider the case where the parameters satisfy certain algebraic relations. By applying appropriate transformations to both the independent variable z and the dependent variable w, we identify conditions under which the transcendental nature of the solution simplifies. This approach allows for the construction of explicit solutions for

W W –1 w –Z w = •

2(2-1) (w –z) a, p, Riccati constants. Following the remarks in work [?], we seek an equation of the form:

$$w' = a(z)w^2 + b(z)w + c(z), \quad (2)$$

all solutions of which are also solutions to equation (1). We shall demonstrate that, under certain relations between j and S, equation (1) possesses a one-parameter family of solutions that constitutes the general solution of a specific Riccati equation. By substituting (2) into (1) and comparing the coefficients of identical powers of w, we obtain:

$$z^2(z-1)^2a^2(z) = 2a,$$

$[z^2(z-1)^2a'(z) - z(z-1)(z+1)a(z)] + z(z-1)(2z-1)a(z) + 2a(z)(z+1) = 0,$

$z^2(z-1)^2[b'(z)+2a(z)c(z)]-z(z-1)(z+1)b(z)-z(z-1)a'(z)+2(z+1)b(z)+2a(z)(z+1)-1[(2z-1)b(z)+2a(z)c(z)]$

- 4z + 1) + 2p z + 2 T (z –1) + 25 z (z –1) = 0,

(z) –(z –1) [(2z –1) (1 –2z –6 (z) –2a (z –2 p (z + 1) –2 (z –1) –25(z –1) = 0 , (z) –2 (z 2za (z) (z) –(z)] (z –1) [(1 –2z–z 1) = 0, (z + (z) 2 (z 1) - 0,

$(z-1)^2c^2(z) + 2\epsilon = 0.$

From the first and last equations of the system, we respectively find $f/2a$.

c(z) =

2 (2 - 1) Painlevé found the general solution of the equation in the case where $f = 6$ [?].

The equalities (3) transform the second and sixth equations of system (A) into identities. The third and fifth equations of system (A) are Riccati equations with respect to $b(z)$, specifically:

$$2z - 1 + 2ac - 3a^2z + 2a'(z+1) + z(z-1)2b' + b^2 + 2z - 1 + 4z + 2b(z-1)$$
$$(z-1)^2(z-1)(z+1) \cdot 2zac + 3c^2 + 2(z+1)c' - 2(z-1)b(z) + 4z + z(z-1)$$

- $= 0.$ (5)

The fourth equation is linear with respect to $z(z-1)$. By rearranging the terms, we obtain:

$$2za + 2(z-1)(2z-1)c - 2(z-1)\dots$$

We seek the general form of the function $\phi(z)$ that reduces equations (4)-(6) to identities. It should be noted that if $\phi(z)$ is a solution to equations (4) and (5), then $\phi(z)$ also satisfies the following linear differential equation:

$$2b(z)[(z+1)(\sqrt{2a} - f - 2ft) - 3z + 1]/\sqrt{2a} = 2(a + ft)(z+1)$$

Furthermore, it identically vanishes the quadratic trinomial:

$$+\cdots - \frac{(\sqrt{2a} - z + \alpha)(\dots)}{(z-1)(ft-a)(z+1)(z-1)\sqrt{2a}}$$

It is easily established that $\phi(z)$ will satisfy equation (5) provided that $\sqrt{2a} - (a + ft + \dots)$.

$b(z) = \dots$

$\sqrt{2a} - f - 2ft/\sqrt{2ft} - (a + ft - 5)/\sqrt{2a}$

/ 2 ^ –/ ^ 2 f t –1 = £ 0 . (9)

By direct substitution of (9) into (7), we verify that $b(z)$ as defined in (9) is indeed a solution to equation (7). Since (9) was derived by eliminating $b'(z)$ from equations (6) and (7), the expression in (9) also satisfies equation (6). It remains to determine the conditions under which (9) causes (8) to vanish identically.

LUKASHEVICH, YABLONSKII. By substituting and equating the coefficients of like powers to zero, we obtain the following relations: $(\mu_k + f = \Gamma_2 p_i) - f_{2o} - p-$.

$2Xp_i + X(f_k + K = \gamma + 1) - \gamma(f_k + 1 = 2\gamma - 1) +$

$f = \gamma - f = -\gamma f_{2a} + 2(- - 5) = 0$, $(f_{2a} + f = 2p + f\mu_{\gamma p} + a-$. It is directly evident that $+f_k + f = 2p = 0$. By summing equations (10)-(10), we obtain:

$(X + \mu + f_{2a} + f- = 2p)^2 = 0,$

That is, equations (10)-(10) are dependent by virtue of the given conditions. Furthermore, the difference between the terms yields $(X + \mu + f_{2a} + f)$, which also vanishes. Substituting the values of X and μ into (10), we obtain after simplification: $a + 3p \approx m - (3a) + 4f - ab(-a - f - 1) + 2(a - b - 6ab - 2a + 2p - 2p5+)$.

$2 - f_{2fS} + \sigma^* = 0,$ (12)

Moreover, the value taken is the one that coincides with $\sqrt{2\alpha - 2\rho}$, as the values of $\sqrt{2\alpha - 2\rho}$ are pre-selected. This condition ensures the existence of a general form for the function $b(z)$ that satisfies equations (4)-(6). Consequently, the equation possesses families of solutions that constitute general solutions to the equations for any choice of the radical values $\sqrt{2\alpha}$ and $\sqrt{...}$ (taken according to (9)), provided that relation (12) is satisfied. We now transition from these equations to linear equations. By setting $2(2W) = ...$, we obtain:

$-\sigma = 0.$ (15) $\frac{2}{(2-1)}\frac{2}{(2-I)^2}$

The equations belong to the Fuchsian class with singular points at $z = 1$ and are characterized by the Riemann P-function:

$$v = P\begin{Bmatrix} 0 & 1 & \infty & \\ -f & -2\beta & 0 & z \\ 2 & ... & ... & \end{Bmatrix}$$

By substituting $v = z^{f+2\alpha}$ ..., equation (15) is reduced to the Gauss hypergeometric equation with arbitrary parameters $\alpha_1, \beta_1, \gamma_1$. Thus, the following holds:

Theorem. If $w(z)$ is a solution to any hypergeometric equation, then by selecting the parameters a, b, γ, δ according to equations (17), (9), and (12), the function $w(z)$ is a solution to the sixth Painlevé equation.

Remark.

1. In the transition from (13) to (15), it was assumed

In the case where $a = 0$, equation (13) yields the linear equation: $2(2-1)/-2p+1/-2ft+1$.

Remark 2. In equation (9), it was assumed that the radicals in (3) were chosen such that they are non-zero. If we assume that $2ft-1=0$, then for the function (2) to exist, it is necessary that the numerators 1 and p also vanish, which leads to the condition:

$m + b = ...$ (22)

In this case, equations (6) and (7) coincide. By integrating one of them, we again obtain:

$b(r) = ...$ (23)

However, in this context, the integration constant is an arbitrary value that must be determined from the equation $f = 2ft$. All arguments regarding equations (8) and (10)-(10') remain valid if conditions (21) and (22) are taken into account, since relation (11) still holds in this case. Consequently, if (12) is replaced by (21) and (22) is determined from (10'), we obtain the solution in the form of (23). These final arguments are also applicable to the case where $2ft+1=0$ in equation (20), as condition (21) holds there as well.

References

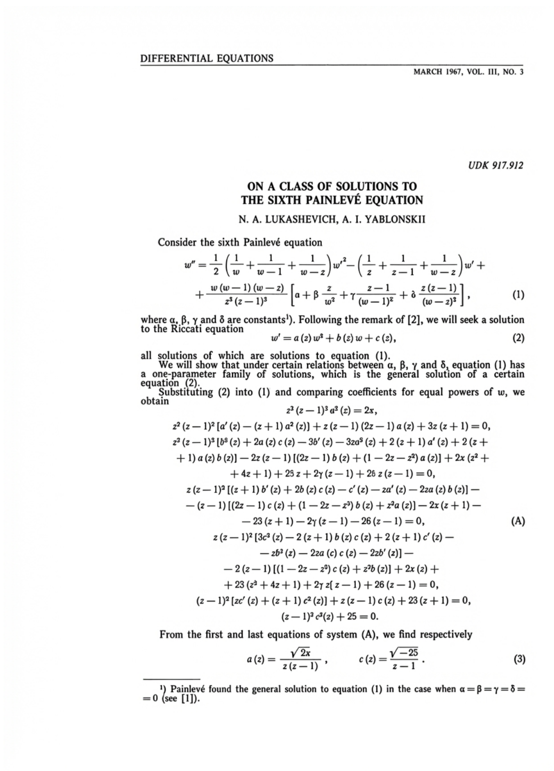
References

Ince, E. L. *Ordinary Differential Equations.* GNTIU, Kharkov, 1939.

Erugin, N. P. *Doklady Akademii Nauk BSSR*, Vol. II, No. 4, pp. 139-142, 1958.

Received by the Editorial Board: December 20, 1966. **Affiliation:** V. I. Lenin Belarusian State University, Institute of Mathematics.

Figures



DIFFERENTIAL EQUATIONS

MARCH 1967, VOL. III, NO. 3

UDK 917.912

ON A CLASS OF SOLUTIONS TO THE SIXTH PAINLEVÉ EQUATION

N. A. LUKASHEVICH, A. I. YABLONSKII

Consider the sixth Painlevé equation

$$w'' = \frac{1}{2}\left(\frac{1}{w} + \frac{1}{w-1} + \frac{1}{w-z}\right)w'^2 - \left(\frac{1}{z} + \frac{1}{z-1} + \frac{1}{w-z}\right)w' + \frac{w(w-1)(w-z)}{z^3(z-1)^3}\left[\alpha + \beta\frac{z}{w^2} + \gamma\frac{z-1}{(w-1)^2} + \delta\frac{z(z-1)}{(w-z)^3}\right], \quad (1)$$

where α, β, γ and δ are constants[1]). Following the remark of [2], we will seek a solution to the Riccati equation

$$w' = a(z)w^3 + b(z)w + c(z), \quad (2)$$

all solutions of which are solutions to equation (1).

We will show that under certain relations between α, β, γ and δ, equation (1) has a one-parameter family of solutions, which is the general solution of a certain equation (2).

Substituting (2) into (1) and comparing coefficients for equal powers of w, we obtain

$$\begin{aligned}
&z^3(z-1)^3 a^3(z) = 2x,\\
&z^2(z-1)^2[a'(z) - (z+1)a^2(z)] + z(z-1)(2z-1)a(z) + 3z(z+1) = 0,\\
&z^3(z-1)^3[b^3(z) + 2a(z)c(z) - 3b'(z) - 3za^3(z) + 2(z+1)a'(z) + 2(z+\\
&+1)a(z)b(z)] - 2z(z-1)[(2z-1)b(z) + (1-2z-z^2)a(z)] + 2x(z^2+\\
&+4z+1) + 2\beta z + 2\gamma(z-1) + 2\delta z(z-1) = 0,\\
&z(z-1)^2[(z+1)b'(z) + 2b(z)c(z) - c'(z) - za'(z) - 2za(z)b(z)] -\\
&-(z-1)[(2z-1)c(z) + (1-2z-z^3)b(z) + z^2a(z)] - 2x(z+1) -\\
&-2\beta(z+1) - 2\gamma(z-1) - 2\delta(z-1) = 0, \qquad (A)\\
&z(z-1)^2[3c^3(z) - 2(z+1)b(z)c(z) + 2(z+1)c'(z) -\\
&-zb^3(z) - 2za(c)c(z) - 2zb'(z)] -\\
&-2(z-1)[(1-2z-z^3)c(z) + z^2b(z)] + 2x(z) +\\
&+2\beta(z^3+4z+1) + 2\gamma z(z-1) + 2\delta(z-1) = 0,\\
&(z-1)^2[zc'(z) + (z+1)c^2(z)] + z(z-1)c(z) + 2\beta(z+1) = 0,\\
&(z-1)^3c^3(z) + 2\delta = 0.
\end{aligned}$$

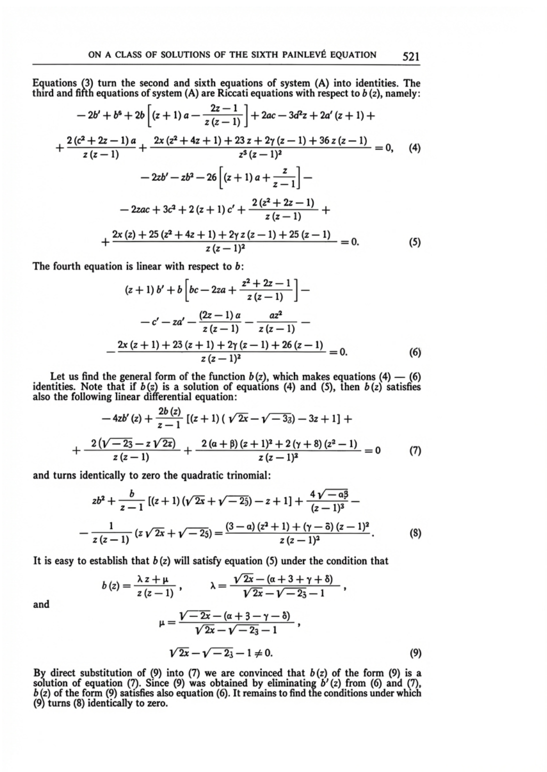
From the first and last equations of system (A), we find respectively

$$a(z) = \frac{\sqrt{2x}}{z(z-1)}, \qquad c(z) = \frac{\sqrt{-2\delta}}{z-1}. \quad (3)$$

[1]) Painlevé found the general solution to equation (1) in the case when $\alpha = \beta = \gamma = \delta = 0$ (see [1]).

Figure 1: Figure 1

Source: RussiaRxiv —Machine translation. Verify with original.



Equations (3) turn the second and sixth equations of system (A) into identities. The third and fifth equations of system (A) are Riccati equations with respect to $b(z)$, namely:

$$-2b' + b^6 + 2b\left[(z+1)a - \frac{2z-1}{z(z-1)}\right] + 2ac - 3d^2z + 2a'(z+1) +$$

$$+\frac{2(c^2+2z-1)a}{z(z-1)} + \frac{2x(z^2+4z+1)+23z+2\gamma(z-1)+36z(z-1)}{z^3(z-1)^2} = 0, \quad (4)$$

$$-2zb' - zb^2 - 26\left[(z+1)a + \frac{z}{z-1}\right] -$$

$$-2zac + 3c^2 + 2(z+1)c' + \frac{2(z^2+2z-1)}{z(z-1)} +$$

$$+\frac{2x(z)+25(z^2+4z+1)+2\gamma z(z-1)+25(z-1)}{z(z-1)^2} = 0. \quad (5)$$

The fourth equation is linear with respect to b:

$$(z+1)b' + b\left[bc - 2za + \frac{z^2+2z-1}{z(z-1)}\right] -$$

$$-c' - za' - \frac{(2z-1)a}{z(z-1)} - \frac{az^2}{z(z-1)} -$$

$$-\frac{2x(z+1)+23(z+1)+2\gamma(z-1)+26(z-1)}{z(z-1)^2} = 0. \quad (6)$$

Let us find the general form of the function $b(z)$, which makes equations (4) — (6) identities. Note that if $b(z)$ is a solution of equations (4) and (5), then $b(z)$ satisfies also the following linear differential equation:

$$-4zb'(z) + \frac{2b(z)}{z-1}\left[(z+1)(\sqrt{2x} - \sqrt{-3\mathfrak{z}}) - 3z + 1\right] +$$

$$+\frac{2(\sqrt{-2\mathfrak{z}} - z\sqrt{2x})}{z(z-1)} + \frac{2(\alpha+\beta)(z+1)^2 + 2(\gamma+8)(z^2-1)}{z(z-1)^3} = 0 \quad (7)$$

and turns identically to zero the quadratic trinomial:

$$zb^2 + \frac{b}{z-1}\left[(z+1)(\sqrt{2x} + \sqrt{-2\mathfrak{z}}) - z + 1\right] + \frac{4\sqrt{-\alpha\mathfrak{z}}}{(z-1)^3} -$$

$$-\frac{1}{z(z-1)}(z\sqrt{2x} + \sqrt{-2\mathfrak{z}}) = \frac{(3-\alpha)(z^2+1) + (\gamma-\delta)(z-1)^2}{z(z-1)^3}. \quad (8)$$

It is easy to establish that $b(z)$ will satisfy equation (5) under the condition that

$$b(z) = \frac{\lambda z + \mu}{z(z-1)}, \qquad \lambda = \frac{\sqrt{2x} - (\alpha + 3 + \gamma + \delta)}{\sqrt{2x} - \sqrt{-2\mathfrak{z}} - 1},$$

and

$$\mu = \frac{\sqrt{-2x} - (\alpha + 3 - \gamma - \delta)}{\sqrt{2x} - \sqrt{-2\mathfrak{z}} - 1},$$

$$\sqrt{2x} - \sqrt{-2\mathfrak{z}} - 1 \neq 0. \quad (9)$$

By direct substitution of (9) into (7) we are convinced that $b(z)$ of the form (9) is a solution of equation (7). Since (9) was obtained by eliminating $b'(z)$ from (6) and (7), $b(z)$ of the form (9) satisfies also equation (6). It remains to find the conditions under which (9) turns (8) identically to zero.

Figure 2: Figure 2

Substituting (9) into (8) and equating the coefficients of like powers of z to nero, we obtain the following relations:

$$\lambda^5 + \lambda(\sqrt{2\alpha} + \sqrt{-2\beta} - 1) - \sqrt{2\alpha} + \alpha - \beta - \gamma + 8 = 0, \qquad (10_1)$$

$$2\lambda\mu + \lambda(\sqrt{2\alpha} + \sqrt{-2\beta} + 1) + \mu(\sqrt{2\alpha} + \sqrt{-2\beta} - 1) + + 4\sqrt{-x^3} - \sqrt{-2\beta} + \sqrt{2\alpha} + 2(\gamma - \delta) = 0, \qquad (10_2)$$

$$\mu^2 + \alpha(\sqrt{2\alpha} + \sqrt{-2\beta} + 1) + \sqrt{-2\beta} + \alpha - \beta - \gamma + \delta = 0. \qquad (10_3)$$

From (9) it is immediately evident, that

$$\lambda + \mu + \sqrt{2\alpha} + \sqrt{-2\beta} = 0, \qquad (11)$$

Comparing equations (10_1) — (10_3), we obtain

$$(\lambda + \mu + \sqrt{2\alpha} + \sqrt{-2\beta})^2 = 0,$$

i.e. equations (10_J) — (10_3) are dependent by virtue of (9). Furthermore, the difference between (10_1) and (10_3) gives

$$(\lambda - \mu - 1)(\lambda + \mu + \sqrt{2\alpha} + \sqrt{-2\beta}),$$

which also vanishes by virtue of (9). Substituting the values of λ and μ from (9) into (10_1), after simplification we obtain

$$2\sqrt{2\alpha}(-\alpha + 3\beta + \gamma - \delta) + 2\sqrt{-2\beta}(3\alpha - \beta - - \gamma + \delta) + 4\sqrt{-\alpha\beta}(-\alpha + \beta + \gamma - 8 - 1) + 2(\alpha - \beta - \gamma) + + x^2 - 6x\beta - 2x\gamma + 2x8 + \beta^3 + 2\beta\gamma - 2\beta\delta + \gamma^3 + 2\gamma\beta + 2^2 = 0, \qquad (12)$$

where the value $2\sqrt{-\alpha\beta}$ is taken to be that which coincides with $\sqrt{2\alpha}\sqrt{-2\beta}$, since the values $\sqrt{2\alpha}$ and $\sqrt{-2\beta}$ and are chosen beforehand. Condition (12) ensures the existence of the general form of the function $b(z)$, satisfying equations (4) — (6).

Thus, equation (1) yields families of solutions which are general relations of the equations

$$\omega' = \frac{\sqrt{2x}}{z(z-1)}\omega^2 + \frac{\lambda\alpha + \mu}{z(z-1)}\omega + \frac{\sqrt{-2\beta}}{z-1} \qquad (13)$$

for any choice of the values of the radicals $\sqrt{2\alpha}$ and $\sqrt{-2\beta}$ (λ and μ are them according to (9)), recurr relation (12) is satisfied.

Let us pass from equation (13) to linear is linear equations. Setting

$$\omega = -\frac{z(z-1)}{\sqrt{2x}}\frac{v'}{v}, \qquad \alpha \neq 0, \qquad (14)$$

we obtain

$$v'' + \frac{(2-\lambda)z - \mu - 1}{z(z-1)}v' + \frac{2\sqrt{-\alpha\beta}}{z(z-1)^2}v = 0. \qquad (15)$$

Equation (15) is an equation of the Fuchsian class, with singular points $z = 0$, $z = 1$, $z = \infty$ and Riemann P-symbol:

$$v = P\begin{Bmatrix} 0 & 1 & \infty & \\ 0 & -\sqrt{-2x} & 0 & z \\ -\mu & -\sqrt{+2x} & 1-\lambda & \end{Bmatrix}. \qquad (16)$$

By the substitution

$$\tau = \frac{1}{1-z}, \qquad \alpha_1 = +\sqrt{2x}, \qquad \beta_1 = +\sqrt{-2\beta}, \qquad \gamma_1 = \lambda \qquad (17)$$

(15) reduces to the Gaussian equation with parameters α_1, β_1 and γ_1

$$\tau(\tau - 1)\sigma''_{s\tau} + [(1 + \alpha_1 + \beta_1)\tau - \gamma_1]\sigma'_z + \alpha_1\beta_2 v = 0. \qquad (18)$$

Thus, the following is valid

Figure 3: Figure 3

Theorem. If $v(\tau)$ is a solution of any hypergeometric equation, then upon choosing parameters α, β, γ, δ, according to (17), (9) and (12),

$$w(z) = \frac{z(z-1)}{\sqrt{2\alpha}} \frac{v'\left(\frac{1}{1-z}\right)}{v\left(\frac{1}{1-z}\right)} \quad (19)$$

is a solution of the sixth Painlevé equation.

Remark 1. Under transition from (13) to (13) it is supposed $\alpha \neq 0$, in case $\alpha = 0$ (13) gives a linear equation

$$w' = \frac{\lambda z + \mu}{z(z-1)} w + \frac{\sqrt{-2\beta}}{z-1} \quad (20)$$

with

$$\lambda = \frac{3 + \gamma + 8}{\sqrt{-2\beta} + 1}, \qquad \mu = \frac{-\sqrt{-2\beta} + (8 - \gamma - 8)}{\sqrt{-2\beta} + 1}.$$

Remark 2. In (9) it was assumed that radicals in (3) are taken such that $\sqrt{2\alpha} - \sqrt{-2\beta} - 1 \neq 0$. If we assume that

$$\sqrt{2\alpha} - \sqrt{-2\beta} - 1 = 0, \quad (21)$$

then for the existence of the function $b(z)$ it is necessary that the numerators for λ and μ vanish, which leads to the condition

$$\gamma + 8 = \frac{1}{2}, \quad (22)$$

and equations (6) and (7) coincide. By integrating one of them, we obtain again

$$b(z) = \frac{\lambda z + \mu}{z(z-1)}, \quad (23)$$

but now λ is an arbitrary constant, which must be found from the equation $\mu = -\lambda - 2\sqrt{-3\beta} - 1$.

All reasoning regarding (8), $(10_1) - (10_3)$ are preserved if (21) and (22) are taken into account, as in this case also (11) takes place. From here, if in (12) we replace (21) and (22) and find λ from (10_1), then we get $b(z)$ in the form (23).

The latter reasoning is also valid in the case when in (20) $\sqrt{-2\beta} + 1 = 0$, as then (21) takes place.

Literature

1. Ains E. L. Ordinary differential equations. GNTIU, Kharkov, 1939.
2. Erugin N. P. DAN BSSR, v. II, No 4, 139—142, 1958.

Received in the editorial office 20 December 1966 g.

Belorussian State University named after V. I. Lenin,
Institute of Mathematics AS BSSR

Figure 4: Figure 4