ON SOME CRITERIA FOR NONOSCILLATION OF LINEAR DIFFERENTIAL OPERATORS

MATHEMATICS

1967

SovietRxiv

Abstract

Full Text

UDC 517.941

MATHEMATICS

G. S. ZAITSEVA

ON SOME CRITERIA FOR NONOSCILLATION OF LINEAR DIFFERENTIAL OPERATORS

(Presented by Academician A. Yu. Ishlinskii, May 15, 1967)

The interval $[a, b]$ is an interval of nonoscillation of the operator

$$L(x) \equiv x^{(n)} + p_1(t)x^{(n-1)} + \cdots + p_n(t)x,$$

if every nontrivial solution of the equation $L[x] = 0$ has on the interval $[a, b]$ no more than $(n-1)$ zeros (zeros are counted according to their multiplicities).

1. **Theorem 1.** Let $|p_i(t)| \le L_i,\ i = 1, 2, \dots, n\ (a \le t \le b)$. If the inequality

$$\sum_{k=2}^{n} \frac{L_k}{\left[\frac{k-1}{2}\right]!\left[\frac{k}{2}\right]! L_1^k} \int_0^{L_1(b-a)/2} \tau^{k-1} e^{\tau}\, d\tau \le 1, \tag{1}$$

is satisfied, then the interval $[a, b]$ is an interval of nonoscillation of the operator

$$L(x) = x^{(n)} + p_1(t)x^{(n-1)} + \cdots + p_n(t)x.$$

Inequality (1) determines a larger interval of nonoscillation than inequality (5)

$$\sum_{k=2}^{n} \frac{L_k (b-a)^k}{k\left[\frac{k-1}{2}\right]!\left[\frac{k}{2}\right]! 2^k} \exp\left\{L_1 \frac{b-a}{2}\right\} \le 1, \tag{2}$$

and therefore refines the estimates ([1–2,5]).

In the proof of Theorem 1 the following is used.

Theorem 2. Let an n-times differentiable function satisfy the conditions

$$x(a_1) = x'(a_2) = \cdots = x^{(n-1)}(a_n) = 0 \qquad (a \le a_1 \le a_2 \le \cdots \le a_n \le b),$$

$$|x^{(n)} + p_1(t)x^{(n-1)}| \le \mu, \qquad |p_1(t)| \le L_1 \qquad (a \le t \le b).$$

Then the inequality holds

$$|x(t)| < \mu \int_0^{L_1(b-a)} \tau^{n-1} e^{\tau}\, d\tau \Big/ L_1^n \left[\frac{n-1}{2}\right]! \left[\frac{n}{2}\right]!.$$

Lemma. If $x(t)$ on $[a, b]$ is a solution of the equation

$$x^{(n)} + p_1(t)x^{(n-1)} = f(t),$$

$$|f(t)| \le \mu, \qquad |p_1(t)| \le L_1 \qquad (a \le t \le b),$$

$$x^{(n-1)}(c) = x^{(n-2)}(c) = \dots = x^{(k)}(c) = 0 \qquad (a \le c \le b),$$

$$k = 0, 1, \dots, n-1,$$

then

$$|x^{(k)}(t)| \le \frac{\mu}{L_1^{n-k}} \left\{ \exp[L_1|t-c|] - 1 - L_1|t-c| - \dots - \frac{[L_1|t-c|]^{n-k-1}}{(n-k-1)!} \right\}.$$

2. Let us note that inequalities (1) and (2) allow the coefficient L_1 to take arbitrarily large values at the expense of the smallness of the remaining co-

coefficients. The following theorem does not depend on the existing criteria (1–5), but it shows that the coefficient $p_2(t)$ may assume arbitrarily large negative values in absolute value, owing to the smallness of the remaining coefficients.

Theorem 3. Let

$$|p_i(t)| \le L_i, \qquad i = 1, 3, 4, \dots, n,$$

$$-L_2 \le p_2(t) \le L_2^+ \qquad (a \le t \le b).$$

The interval $[a, b]$ of length $b - a < 2\min[h_1, h_2]$ is an interval of nonoscillation of the operator

$$L(x) = x^{(n)} + p_1 x^{(n-1)} + \cdots + p_{nx},$$

if h_1, h_2 are, respectively, the first positive roots of the equations

$$\sum_{k=3}^{n} \frac{L_k h_1^k}{(k-2)\left[\frac{k-3}{2}\right]!\left[\frac{k-2}{2}\right]!} = F_1(h_1), \qquad \sum_{k=3}^{n} \frac{L_k h_2^k}{(k-2)\left[\frac{k-3}{2}\right]!\left[\frac{k-2}{2}\right]!} = F_2(h_2),$$

where

$$F_1(h) = \frac{L_2}{\left[\operatorname{ch}\eta h - \frac{L_1}{2\eta}\operatorname{sh}\eta h\right]\exp\frac{L_1}{2}h - 1}, \qquad \eta = \sqrt{\frac{L_1^2}{4} + L_2};$$

$$F_2(h) = \begin{cases} \dfrac{L_2^+\left[\operatorname{ch}\beta_1 h - \frac{L_1}{2\beta_1}\operatorname{sh}\beta_1 h\right]\exp\frac{L_1}{2}h}{1 - \left[\operatorname{ch}\beta_1 h - \frac{L_1}{2\beta_1}\operatorname{sh}\beta_1 h\right]\exp\frac{L_1}{2}h}, & \text{if } \beta_1^2 = \dfrac{L_1^2}{4} - L_2^+ > 0, \\ \dfrac{L_2^+\left[\cos\beta_2 h - \frac{L_1}{2\beta_2}\sin\beta_2 h\right]\exp\frac{L_1}{2}h}{1 - \left[\cos\beta_2 h - \frac{L_1}{2\beta_2}\sin\beta_2 h\right]\exp\frac{L_1}{2}h}, & \text{if } \beta_2^2 = L_2^+ - \dfrac{L_1^2}{4} > 0, \\ \dfrac{L_2^+\left[1 - \sqrt{L_2^+}\,h\right]\exp\sqrt{L_2^+}\,h}{1 - \left[1 - \sqrt{L_2^+}\,h\right]\exp\sqrt{L_2^+}\,h}, & \text{if } L_2^+ = \dfrac{L_1^2}{4}, \\ \dfrac{L_1^2}{\exp L_1 h - 1 - L_1 h}, & \text{if } L_2^+ = 0, \\ \dfrac{2}{h^2}, & \text{if } L_2^+ = L_1 = 0. \end{cases} \tag{3}$$

For $n = 2$ the stated nonoscillation criterion coincides with the criterion, unimprovable in the characteristics L_1, L_2^+ (6).

In the proof of Theorem 3 the following is used.

Lemma. Let the interval $[a, b]$ be an interval of nonoscillation of the operator

$$L[x] = x'' + p_1 x' + p_2 x,$$

$$|p_1(t)| \le L_1, \qquad p_2(t) \le L_2^+ \qquad (a \le t \le b),$$

and let the function $v(t)$ be a solution of the problem

$$Lv = 1, \qquad v(a) = v(b) = 0.$$

Then

$$|v(t)| \leq \frac{1}{F_2[(b-a)/2]} \qquad (a \leq t \leq b),$$

where $F_2(h)$ is defined by the inequalities (3).

Bauman Higher Technical School
Received
15 V 1967

CITED LITERATURE

[1] D. Zh. Sansone, *Ordinary Differential Equations*, Moscow–Leningrad, 1953.
[2] A. Yu. Levin, *Mathematical Collection*, 64 (106), 3, 396 (1964).
[3] G. A. Bessmertnykh, A. Yu. Levin, *DAN*, 144, No. 3, 471 (1962).
[4] A. Yu. Levin, *DAN*, 153, No. 6 (1963).
[5] G. S. Zaitseva, *DAN*, 176, No. 4 (1967).
[6] H. Erpheser, *Math. Zs.*, 61, No. 4 (1955).

Note: Figure translations are in progress. See original paper for figures.