CORRECT SPACES

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V. A. EFREMOVICH, A. G. MORDKOVICH, V. Yu. SANDBERG

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By a **general uniform space** we mean here a set X in which a system $\mathfrak{U}$ of symmetric, reflexive relations (“orders of closeness”) between elements $x \in X$ is defined, satisfying the single condition: if the symmetric relation $v < u,\ u \in \mathfrak{U}$, then $v \in \mathfrak{U}$ (the relation $v < u$ means that from xuy it follows that xvy for any $x, y \in X$). In what follows, by a **uniform space** we mean a general uniform space satisfying the triangle axiom: for every $u \in \mathfrak{U}$ there exists a $v \in \mathfrak{U}$ such that $v^2 < u$ (xv^2y means that there exists a point $z \in X$ for which $xvz,\ zvy$).

It is natural, among all uniform spaces, to single out the class of spaces generating a full-fledged closeness, i.e., a closeness satisfying all the usual axioms (see, for example, axioms 0, 1, 2, 3 in ([1])). Such uniform spaces we shall call **correct**.* The following criterion of correctness is easily derived:

For a uniform space to be correct it is necessary and sufficient that two conditions be satisfied:

a) **Separability.** If $x, y \in X$ and $x \neq y$, then there exists a $u \in \mathfrak{U}$ such that $x\bar{u}y$.

b) **The condition of intersection of neighborhoods.** If $M \subset X,\ u, v \in \mathfrak{U}$, then there exists a $w \in \mathfrak{U}$ such that $M_w \subset M_u \cap M_v$ (by M_u is denoted the u-neighborhood of the set M), i.e., the intersection of two neighborhoods of any set is again its neighborhood.

Of some interest are also somewhat more general spaces; we shall call them **semicorrect**. These are uniform spaces satisfying condition a) and

b′) The intersection of two neighborhoods of any point is again its neighborhood.

1. Completeness

We shall say that a general uniform space X is **complete** in a certain class K of spaces if X belongs to the class K and it cannot be densely embedded in a space of this class distinct from X. A **completion** of X in K is a space $\widetilde{X}$ complete in this class and containing X as an everywhere dense subspace. In what follows, the indication of the class is contained in the name of the space.

For example, a correct space is complete if it cannot be densely embedded in a correct space distinct from it.

Theorem 1. *In the class of separable uniform spaces there are no complete spaces.*

Proof. Let $(X, \mathfrak{U})$ be a separable uniform space. Put $\widetilde{X} = X \cup a$, where $a \notin X$, and define on $\widetilde{X}$ the structure $\widetilde{\mathfrak{U}}$: to each pair (u, z), where $u \in \mathfrak{U},\ z \in X$, assign the relation $u(z)$, and for $x, y \in X$ put $xu(z)y$ if and only if xuy;

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* Here the term correct space is used in a different sense than in paper ([2]). The spaces considered there should now logically be renamed **general correct**.

$xu(z)a$, if and only if xuz; $au(z)a$ for every $u(z)$. It is easy to verify that the system $\mathbf{U}$ of relations $u(z)$ turns $\widetilde{X}$ into a separated uniform space containing X as a dense subspace.

Thus, we see that an attempt to abandon completely Weil' s intersection axiom leads to an unsuccessful class of spaces. On the other hand, the class of Weil spaces is somewhat narrow: for example, it does not always contain a maximal structure compatible with the given closeness, which leads to difficulties in defining completeness of proximity spaces, their products, etc. At the same time, correct spaces admit a full-fledged theory of completion (this is the content of item 2) and they are in good agreement with closeness (this is the content of item 3)*.

2. Completion. A Cauchy filter $\mathcal{F}$ in a correct space X is called an **infrafilter** if there is no Cauchy filter $\mathcal{F}_1$ that is essentially weaker, i.e. such that $\mathcal{F}_1 \subset \mathcal{F}$, $\mathcal{F}_1 \neq \mathcal{F}$. Denote by $\mathbf{U}(\mathcal{F})$ the filter consisting of all neighborhoods of all elements of the Cauchy filter $\mathcal{F}$. It is easily verified that $\mathbf{U}(\mathcal{F})$ is a Cauchy filter. If $\mathcal{F}$ has a base consisting of a single set $\{x\}$, then $\mathbf{U}(\mathcal{F})$ coincides with the filter $\mathbf{U}(x)$ of neighborhoods of the point $x \in X$.

Lemma 1. *In order that a Cauchy filter $\mathcal{F}$ in a correct space be an infrafilter, it is necessary and sufficient that for every $A \in \mathcal{F}$ there exist $B \in \mathcal{F}$, $u \in \mathbf{U}$, such that $B_u \subset A$.*

Let $\mathcal{F}$ be an infrafilter. Since $\mathbf{U}(\mathcal{F}) \subset \mathcal{F}$, we have $\mathcal{F} = \mathbf{U}(\mathcal{F})$, and for $\mathbf{U}(\mathcal{F})$ the assertion follows at once from the triangle axiom. Conversely, suppose that the conditions of the lemma are satisfied and that $\mathcal{F}_1 \subset \mathcal{F}$ is a Cauchy filter. Take $A \in \mathcal{F}$ and find $B \in \mathcal{F}$ and $u \in \mathbf{U}$ such that $B_u \subset A$. The filter $\mathcal{F}_1$ contains an element C small of order u, and since $C \in \mathcal{F}$, we have $B \cap C \neq \emptyset$ and, consequently, $C \subset B_u \subset A$. Hence $A \in \mathcal{F}_1$, i.e. $\mathcal{F}_1 = \mathcal{F}$, which proves the lemma.

Corollary 1. *The filter $\mathbf{U}(x)$ of neighborhoods of a point $x \in X$ is an infrafilter.*

Corollary 2. *Every infrafilter has a base consisting of open sets.*

Corollary 3. *For every Cauchy filter there exists an infrafilter contained in it (it suffices to take the infrafilter $\mathbf{U}(\mathcal{F})$).*

Lemma 2. *If $\mathcal{F}$ is an infrafilter, then for any $u, v \in \mathbf{U}$ there is $w \in \mathbf{U}$ such that, if $A \in \mathcal{F}$ and A is small of order w, then A is small of order u and small of order v.*

Let $B, C \in \mathcal{F}$, where B is small of order u, and C is small of order v. Since $D = B \cap C \in \mathcal{F}$, there exist $E \in \mathcal{F}$ and $w \in \mathbf{U}$ such that $E_w \subset D$. Now take $A \in \mathcal{F}$ small of order w. Since $A \subset E_w \subset D$, A is small of order u and small of order v.

Remark. In a semicorrect space this lemma loses its force, as the following example shows. As X take the real line without zero; to each $\varepsilon > 0$ assign two relations u_ε and $u_{-\varepsilon}$, defined by the conditions: $x u_\varepsilon y \Longleftrightarrow ((x < \varepsilon) \wedge (y < \varepsilon)) \vee (|x - y| < \varepsilon)$; $x u_{-\varepsilon} y \Longleftrightarrow ((x > -\varepsilon) \wedge (y > -\varepsilon)) \vee (|x - y| < \varepsilon)$. It is easy to verify that the structure $\mathbf{U} = \{u_\varepsilon, u_{-\varepsilon}\}$ is semicorrect, and for the Cauchy filter with base consisting of the intervals $(-a, a)$, $a > 0$, Lemma 2 is not true. A Cauchy filter in a semicorrect space for which Lemma 2 holds will be called **proper**.

Lemma 3. *If a correct space $(X, \mathbf{U})$ is densely embedded in $(\widetilde{X}, \widetilde{\mathbf{U}})$, then every infrafilter $\widetilde{\mathcal{F}}$ in $\widetilde{X}$ induces an infrafilter $\mathcal{F}$ in X.*

Lemma 4. *If in a correct space X every infrafilter is the neighborhood filter of some point, then X is complete.*

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* A somewhat narrower class of spaces was considered in paper ([6]).

Suppose the contrary. Then X can be densely embedded in $\widetilde{X} \neq X$. Let $x_0 \in \widetilde{X} \setminus X$ and $\widetilde{\mathcal{F}} = \tilde{\mathbf{u}}(x_0)$. Since $\widetilde{\mathcal{F}}$ is an infrafilter, it induces in X an infrafilter $\mathcal{F}$, which must coincide with $\mathbf{u}(x)$ for some point $x \in X$. But then $x \in \bigcap \widetilde{\mathcal{F}}$ and $x_0 \in \bigcap \widetilde{\mathcal{F}}$, which is impossible in view of the separability of $\widetilde{X}$.

Lemma 5. *If a correct space X is densely embedded in a semicorrect space $\widetilde{X}$, then $\widetilde{X}$ is correct.*

It is necessary to prove that the intersection S of two neighborhoods S_1 and S_2 of a set $M \subset \widetilde{X}$ is also a neighborhood of this set. Since every neighborhood of the set M is a neighborhood of some open set containing M, and the intersection of two open sets in a semicorrect space is open, without loss of generality one may assume that M is open, while S_1 and S_2 are the open kernels of closed sets. The set $M \cap X$ is nonempty and $S \cap X$ is its neighborhood in X, i.e. $M \cap X \, \bar{\delta} (X \setminus S \cap X)$. Then $\overline{M \cap X} \, \delta (\overline{X \setminus S \cap X})$ (the closure is taken in $\widetilde{X}$). But since $M \subset \overline{M \cap X}$, and $(X \setminus S \cap X) = \widetilde{X} \setminus S$, it follows that $M \, \bar{\delta} (\widetilde{X} \setminus S)$, which proves the lemma.

Theorem 2. *Every correct space X has a unique (up to a uniformorphism leaving the points of the space X fixed) completion.*

Proof. Consider the set $\widetilde{X}$ of all infrafilters of the space X, and define in the following way on $\widetilde{X}$ a general uniform structure $\tilde{\mathbf{u}}$: for $\alpha, \beta \in \widetilde{X}$ put $\alpha \tilde{u} \beta$ if and only if α and β have a common element of small order $u \in \mathbf{u}$. In particular, if $\alpha = \mathbf{u}(x)$, $\beta = \mathbf{u}(y)$ $(x, y \in X)$, then from $\alpha \tilde{u} \beta$ there follows xuy, and from xuy, $v^2 \succ u$ there follows $\alpha \tilde{v} \beta$. This means that the mapping $x \to \mathbf{u}(x)$ is an embedding of X into $\widetilde{X}$, and we may identify x and $\mathbf{u}(x)$. Let us prove the correctness of $\widetilde{X}$.

Let $\alpha \neq \beta$. Suppose that for every $\tilde{u} \in \tilde{\mathbf{u}}$ we have $\alpha \succ \beta$, i.e. α and β have arbitrarily small common elements. But then $\gamma = \alpha \cap \beta$ is a Cauchy filter, and since $\gamma \subset \alpha$, $\gamma \subset \beta$, we have $\gamma = \alpha$, $\gamma = \beta$—a contradiction. Thus $\tilde{\mathbf{u}}$ satisfies condition a).

Let $\tilde{u} \in \tilde{\mathbf{u}}$, $u, v \in \mathbf{u}$, and $v^2 \succ u$. If $\alpha \tilde{v} \beta$, $\beta \tilde{v} \gamma$, then α and β have a common element M of small order v, while β and γ have a common element N of small order v. Then α and γ have the common element $M \cup N$, of small order u, i.e. $\alpha \tilde{u} \gamma$. Hence, $\tilde{v}^2 \succ \tilde{u}$, and consequently $\tilde{\mathbf{u}}$ satisfies the triangle axiom.

Let $\alpha \in \widetilde{X}$, $\tilde{u}, \tilde{v} \in \tilde{\mathbf{u}}$. By Lemma 2, there exists $w \in \mathbf{u}$ such that every element of the filter α of small order w is of small order u and of small order v. Let us verify that $a_{\widetilde{w}} \subset a_{\widetilde{u}} \cap a_{\tilde{v}}$.

Let $\beta \in a_{\widetilde{w}}$. This means that α and β have a common element M, of small order w. But then M is of small order u and of small order v, i.e. $\beta \in a_{\widetilde{u}}$, $\beta \in a_{\tilde{v}}$. Thus $\tilde{\mathbf{u}}$ satisfies condition b', i.e. $\widetilde{X}$ is a semicorrect space.

Let $\alpha \in \widetilde{X}$, $\tilde{u} \in \tilde{\mathbf{u}}$, $\mathcal{G} \in \alpha$, where $\mathcal{G}$ is an open set of small order u. If $x \in \mathcal{G}$, then $\mathcal{G} \in \mathbf{u}(x)$, i.e. $\mathbf{u}(x)$ and α have a common element of small order u. This means that $\mathbf{u}(x)\, \tilde{u}\, \alpha$. Consequently, X is densely embedded in $\widetilde{X}$, and, by Lemma 5, $(\widetilde{X}, \tilde{\mathbf{u}})$ is a correct space.

Finally, let us verify the completeness of the space $\widetilde{X}$. Let $\widetilde{\mathcal{F}}$ be an infrafilter in $\widetilde{X}$, and let $\mathcal{F}$ be the infrafilter induced by it in X. We shall show that $\widetilde{\mathcal{F}} = \tilde{\mathbf{u}}(\mathcal{F})$, whence, by Lemma 4, the completeness of $\widetilde{X}$ will follow. Choose in $\widetilde{\mathcal{F}}$ an open element $\widetilde{\mathcal{G}}$, of small order $\tilde{u} \in \tilde{\mathbf{u}}$. Then $\mathcal{G} = \widetilde{\mathcal{G}} \cap X \in \mathcal{F}$, where $\mathcal{G}$ is open and of small order u. If $x \in \mathcal{G}$, then $\mathcal{G}$ is a common element of the infrafilters $\mathbf{u}(x)$ and $\mathcal{F}$. This means that $\mathcal{F}\, \tilde{u}\, \mathbf{u}(x)$, i.e. $\mathcal{G} \subset \tilde{\mathbf{u}}(\mathcal{F})$. And since every element of the filter $\widetilde{\mathcal{F}}$ has nonempty intersection with $\mathcal{G}$, $\widetilde{\mathcal{F}}$ is a contact point of the filter $\widetilde{\mathcal{F}}$. Hence it follows from (3) that $\widetilde{\mathcal{F}}$ converges to $\mathcal{F}$, i.e. $\tilde{\mathbf{u}}(\mathcal{F}) \subset \widetilde{\mathcal{F}}$. But $\widetilde{\mathcal{F}}$ is an infrafilter; therefore, $\tilde{\mathbf{u}}(\mathcal{F}) = \widetilde{\mathcal{F}}$.

The uniqueness of the completion is proved in the same way as was done in ([3]) for uniform spaces in the sense of A. Weil.

Let us note that, in passing, with the proof of Theorem 2 it has been established that, if a space is complete, then in it every infrafilter is the filter of neighborhoods of some point. Now the following can easily be obtained.

Theorem 2′. *For completeness of a correct space it is necessary and sufficient that every Cauchy filter in it converge.*

Theorem 3. *Every semicorrect space has a unique completion.*

As a completion one takes the set of all regular infrafilters. In all other respects the proof of this theorem does not differ from the proof of Theorem 2.

3. Maximal structure

Theorem 4. *In the set of all correct structures of a given proximity space there is a maximal element.*

$\mathbf{u}_{\max}$ is defined as the union of all admissible correct structures. It is easy to verify that the common uniform structure $\mathbf{u}_{\max}$ determines the given proximity, and therefore is a common correct structure. Since, moreover, $\mathbf{u}_{\max}$ satisfies the triangle axiom, $\mathbf{u}_{\max}$ is a correct structure.

Let (X, δ) be a proximity space and $(X, \mathbf{u}_{\max})$ the maximal admissible correct space.

Definition. The proximity space generated by the completion $(\widetilde{X}, \tilde{\mathbf{u}}_{\max})$ of the space $(X, \mathbf{u}_{\max})$ is called the **completion** of the proximity space (X, δ).

It turns out that this definition coincides with the definition of Yu. M. Smirnov ([4]). The point is that, if one passes to coverings, then $\mathbf{u}_{\max}$ is the totality of all uniform δ-coverings of X; infrafilters in $(X, \mathbf{u}_{\max})$ coincide with the c-ends of the space X defined in ([4]), and the proximity in $\widetilde{X}$ generated by the structure $\tilde{\mathbf{u}}_{\max}$ canonically coincides with the proximity introduced in ([4]) on the set of c-ends.

Definition. Let a family of proximity spaces $(X_\mu)_{\mu \in \mathfrak{M}}$ be given. Their **product** is the set

$$X = \prod_{\mu \in \mathfrak{M}} X_\mu,$$

the proximity relation between inadmissible subsets of which is generated by the product of the correct spaces $\{X_\mu, (\mathbf{u}_\mu)_{\max}\}$, where $(\mathbf{u}_\mu)_{\max}$ is the maximal admissible correct structure of the space X_μ.

One can prove that this definition coincides with the strong product of proximity spaces proposed in ([5]).

The proofs (with one inessential exception) were carried out by A. G. Mordkovich and V. Yu. Sandberg.

Mathematical Institute named after V. A. Steklov
Academy of Sciences of the USSR

Moscow Pedagogical Institute
named after V. I. Lenin

Voronezh University

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Note: Figure translations are in progress. See original paper for figures.