SOME ASYMPTOTIC FORMULAS FOR THE SPECTRA OF THE THIRD BOUNDARY-VALUE PROBLEM RELATED TO VARIATION OF THE FUNCTION ENTERING THE BOUNDARY CONDITION

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Abstract

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SOME ASYMPTOTIC FORMULAS FOR THE SPECTRA OF THE THIRD BOUNDARY-VALUE PROBLEM RELATED TO VARIATION OF THE FUNCTION ENTERING THE BOUNDARY CONDITION

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Let V be a bounded domain in three-dimensional space, and let S be its boundary. Further, let $\{\mu_k(\sigma)\}$ $(\mu_1(\sigma) \leq \mu_2(\sigma) \leq \ldots)$ be the eigenvalues of the operator Lu, generated by the operation $-\Delta u$ under the boundary condition

$$[\partial u/\partial n + \sigma u]_S = 0,$$

where $\partial u/\partial n$ is differentiation in the direction of the outward normal.

It is known (see ([1])) that

$$\mu_n(\sigma) \sim (6\pi^2 v^{-1} n)^{2/3} \qquad (n \to \infty) \tag{1}$$

(v is the volume of the domain V); from this it is not difficult to derive the relation

$$\sum_{\mu_k(\sigma)\leq x} \mu_k(\sigma_1) \sim \frac{v}{10\pi^2} x^{5/2} \qquad (x \to +\infty),$$

where $\sigma(Q)$ and $\sigma_1(Q)$ $(Q \in S)$ are two, generally speaking, different functions (the summation extends over all values of k for which $\mu_k(\sigma) \leq x$).

In this note the asymptotic behavior of the difference

$$\sum_{\mu_k(\sigma)\leq x} \mu_k(\sigma_2) - \sum_{\mu_k(\sigma)\leq x} \mu_k(\sigma_1);$$

is studied; namely, the equality

$$\sum_{\mu_k(\sigma)\leq x} [\mu_k(\sigma_2) - \mu_k(\sigma_1)] = \frac{1}{3\pi^2} \int_S [\sigma_2(Q) - \sigma_1(Q)]\, ds \cdot x^{3/2} + o(x^{3/2}) \qquad (x \to +\infty). \tag{2}$$

is established.

Some of its generalizations are also given.

Formula (2) may be regarded as a generalization to the three-dimensional case of the well-known formula of I. M. Gelfand and B. M. Levitan for the difference of the traces of two Sturm–Liouville operators (see ([2])).

Let Q be an arbitrary point of the surface S. We shall assume that some neighborhood of this point on the surface S can be given by the equation $z = z_Q(x, y)$ (the z-axis is directed along the normal to S at the point Q), and that the function $z_Q(x, y)$ everywhere in the disk $x^2 + y^2 < \rho_0^2$ (where ρ_0 does not depend on Q) has second partial derivatives satisfying a Lipschitz condition (with exponent equal to 1 and with a constant independent of Q).

Theorem 1. *If the function $\sigma(Q)$ satisfies a Lipschitz condition, then as $p \to +\infty$*

$$\sum_{k=1}^{\infty} \frac{1}{[\mu_k(\sigma) + p]^2} = \frac{v}{8\pi} \frac{1}{\sqrt{p}} + \frac{s}{16\pi} \frac{1}{p} + \frac{1}{24\pi} \int_S [h(Q) - 3\sigma(Q)]\, ds\, \frac{1}{p^{3/2}} + O\left(\frac{1}{p^2}\right), \tag{3}$$

where v and s are the volume of the domain V and the area of the surface S; $h(Q)$ is the mean curvature of the surface S at the point Q.

Proof. Let $G(M, M_1, -\chi^2) = (4\pi r_{MM_1})^{-1} e^{-\chi r_{MM_1}} - g(M, M_1, \chi)$ be the Green' s function of the expression $\Delta u - \chi^2 u$ (χ is a sufficiently large positive constant) for the domain V under the boundary condition $[\partial u/\partial n + \sigma u]_S = 0$. As $M_1 \to M$, from the equality

$$G(M, M_1, -\chi_1^2) - G(M, M_1, -\chi^2) = (\chi^2 - \chi_1^2) \sum_{k=1}^{\infty} \frac{u_k(M) u_k(M_1)}{[\mu_k(\sigma) + \chi_1^2][\mu_k(\sigma) + \chi^2]},$$

where $\{u_k(M)\}$ are the eigenfunctions of the operator Lu, and $\chi_1 \neq \chi$, it follows that

$$\frac{\chi_1 - \chi}{4\pi} + g(M, M, \chi_1) - g(M, M, \chi) = (\chi_1^2 - \chi^2) \sum_{k=1}^{\infty} \frac{u_k^2(M)}{[\mu_k(\sigma) + \chi_1^2][\mu_k(\sigma) + \chi^2]}.$$

Hence follows the formula

$$\sum_{k=1}^{\infty} \frac{1}{[\mu_k(\sigma) + \chi^2]^2} = \frac{v}{8\pi} \frac{1}{\chi} + \frac{1}{2\chi} \int_V g'_\chi(M, M, \chi)\, dv_M.$$

By means of a method close to that used in (3), it can be proved that

$$\int_V g'_\chi(M, M, \chi)\, dv_M =$$

$$= \frac{s}{8\pi}\frac{1}{\chi} + \frac{1}{12\pi}\int_S [h(Q) - 3\sigma(Q)]\, ds \cdot \frac{1}{\chi^2} + O\left(\frac{1}{\chi^3}\right) \qquad (\chi \to +\infty).$$

Combining the last two equalities and putting $\chi^2 = p$, we obtain (3).

Theorem 2. Let $\sigma(Q)$, $\sigma_1(Q)$, and $\sigma_2(Q)$ $(Q \in S)$ be any three functions satisfying the Lipschitz condition. Then

$$\sum_{\mu_k(\sigma_2) \le x} [\mu_k(\sigma_2) - \mu_k(\sigma_1)] = \frac{I}{3\pi^2} x^{3/2} + O(x^{3/2}) \qquad (x \to +\infty), \tag{4}$$

where

$$I = \int_S [\sigma_2(Q) - \sigma_1(Q)]\, ds.$$

Proof. Suppose first that $\sigma_2(Q) \ge \sigma_1(Q)$. Introduce for consideration the nonnegative and nondecreasing function

$$\Phi(x) = \sum_{\mu_k(\sigma_2) \le x} [\mu_k(\sigma_2) - \mu_k(\sigma_1)];$$

moreover, set

$$\psi(p) = \sum_{k=1}^{\infty} \frac{[\mu_k(\sigma_2) - \mu_k(\sigma_1)]^2 [\mu_k(\sigma_2) + 2\mu_k(\sigma_1) + 3p]}{[\mu_k(\sigma_2) + p]^3 [\mu_k(\sigma_1) + p]^2}.$$

We now use Theorem 1: let us write formula (4) first for the function $\sigma_1(Q)$, and then for the function $\sigma_2(Q)$; subtracting the second from the first equality, we find that for any $\alpha < \mu_1(\sigma_2)$

$$2\int_\alpha^{+\infty} \frac{d\Phi(x)}{(x+p)^3} + \psi(p) = \frac{I}{8\pi}\frac{1}{p^{3/2}} + O\left(\frac{1}{p^2}\right) \qquad (p \to +\infty).$$

Taking further into account that

$$\psi(p) = o\left[\int_\alpha^{+\infty} (x+p)^{-3}\, d\Phi(x)\right] \qquad \text{as } p \to +\infty,$$

we have

$$\int_{\alpha}^{+\infty} \frac{d\Phi(x)}{(x+p)^3} \sim \frac{I}{16\pi} \frac{1}{p^{3/2}} \qquad (p \to +\infty).$$

Applying the Hardy-Littlewood Tauberian theorem (see, for example, ([4])), we find:

$$\Phi(x) = \sum_{\mu_k(\sigma_2) \le x} [\mu_k(\sigma_2) - \mu_k(\sigma_1)] \sim \frac{I}{3\pi^2} x^{3/2} \qquad (x \to +\infty).$$

Similarly, the asymptotic equality is established

$$\sum_{\mu_k(\sigma_1) \le x} [\mu_k(\sigma_2) - \mu_k(\sigma_1)] \sim \frac{I}{3\pi^2} x^{3/2} \qquad (x \to +\infty).$$

Taking further into account that

$$\sum_{\mu_k(\bar{\sigma}) \le x} [\mu_k(\sigma_2) - \mu_k(\sigma_1)] \le \sum_{\mu_k(\sigma) \le x} \le \sum_{\mu_k(\underline{\sigma}) \le x},$$

where $\bar{\sigma}(Q) = \max\{\sigma(Q), \sigma_2(Q)\}$, $\underline{\sigma}(Q) = \min\{\sigma(Q), \sigma_1(Q)\}$, and that as $x \to +\infty$ the functions

$$\sum_{\mu_k(\bar{\sigma}) \le x} [\mu_k(\sigma_2) - \mu_k(\sigma_1)] =$$

$$= \sum_{\mu_k(\bar{\sigma}) \le x} [\mu_k(\bar{\sigma}) - \mu_k(\sigma_1)] - \sum_{\mu_k(\bar{\sigma}) \le x} [\mu_k(\bar{\sigma}) - \mu_k(\sigma_2)],$$

$$\sum_{\mu_k(\underline{\sigma}) \le x} [\mu_k(\sigma_2) - \mu_k(\sigma_1)]$$

are asymptotically equal to $\frac{I}{3\pi^2} x^{3/2}$, we obtain (4) for the special case $\sigma_2(Q) \ge \sigma_1(Q)$.

To prove the theorem in the general case, it is now enough to apply the equality

$$\sum_{\mu_k(\sigma) \le x} [\mu_k(\sigma_2) - \mu_k(\sigma_1)] = \sum_{\mu_k(\sigma) \le x} [\mu_k(\sigma_0) - \mu_k(\sigma_1)] - \sum_{\mu_k(\sigma) \le x} [\mu_k(\sigma_0) - \mu_k(\sigma_2)],$$

where $\sigma_0(Q) = \max\{\sigma_1(Q), \sigma_2(Q)\}$, and to use the special case of this theorem considered above.

Taking (1) into account, we obtain

Corollary 1. As $n \to \infty$,

$$\sum_{k=1}^{n} [\mu_k(\sigma_2) - \mu_k(\sigma_1)] = \frac{2I}{v} n + o(n).$$

Remark. From Theorem 2 (or Corollary 1) it follows that if the integral

$$I = \int_S [\sigma_2(Q) - \sigma_1(Q)]\, ds \neq 0,$$

then for infinitely many values of the index k the difference $\mu_k(\sigma_2) - \mu_k(\sigma_1)$ has the same sign as the integral I.

With the aid of Theorem 2, the following can easily be proved.

Theorem 3. Let the functions $\sigma(Q)$, $\sigma_1(Q)$, $\sigma_2(Q)$ and the quantity I be the same as in Theorem 2. Suppose, further, that on the half-axis $[a, +\infty)$ there is given a sign-definite function $f(x)$, absolutely continuous on every interval $[a, b]$ $(b < +\infty)$; in addition, assume that the expression $xf'(x)[f(x)]^{-1}$ is bounded almost everywhere and

$$\int^{+\infty} x^{1/2} f(x)\, dx = \infty.$$

Then, as $x \to +\infty$,

$$\sum_{a<\mu_k(\sigma)\leq x} f[\mu_k(\sigma)]\, [\mu_k(\sigma_2) - \mu_k(\sigma_1)] = \left(\frac{I}{2\pi^2} + o(1)\right) \int_a^x |t|^{1/2} f(t)\, dt.$$

Assuming that $f(x) = x^m$ $(m \geq -3/2)$, we obtain

Corollary 2. As $x \to +\infty$,

$$\sum_{0<\mu_k(\sigma)\leq x} \mu_k^m(\sigma)\, [\mu_k(\sigma_2) - \mu_k(\sigma_1)] = \begin{cases} \dfrac{I}{(2m+3)\pi^2}\, x^{m+3/2} + o\big(x^{m+3/2}\big), & (m > -3/2), \\ \dfrac{I}{2\pi^2} \ln x + o(\ln x), & (m = -3/2). \end{cases}$$

From Corollary 2 it is not difficult to derive the following assertions:

A. If $\mu_k(\sigma_1) \neq 0$, then as $n \to \infty$

$$\sum_{k=1}^n \frac{\mu_k(\sigma_2)}{\mu_k(\sigma_1)} = n + \left(\frac{6}{\pi^2 \nu}\right)^{1/3} I n^{1/3} + o\big(n^{1/3}\big).$$

B. If $\mu_1(\sigma_1) > 0$ and $\mu_1(\sigma_2) > 0$, then as $n \to \infty$

$$\sum_{k=1}^n [\mu_k^m(\sigma_2) - \mu_k^m(\sigma_1)] =$$

$$= \begin{cases} \dfrac{m}{(2m+1)\pi^2}\left(\dfrac{6\pi^2}{2}\right)^{(2m+1)/3} In^{(2m+1)/3} + o\left(n^{(2m+1)/3}\right), & (m > -1/2), \\ -\dfrac{I}{6\pi^2}\ln n + o(\ln n), & (m = -1/2). \end{cases}$$

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Note: Figure translations are in progress. See original paper for figures.