ON ALGORITHMIC PROBLEMS FOR SOME CLASSES OF QUASIGROUPS

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Abstract

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ON ALGORITHMIC PROBLEMS FOR SOME CLASSES OF QUASIGROUPS

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By a quasigroup we shall mean a set with three binary operations $A, A^{-1}, {}^{-1}A$, satisfying the identities:

$$1.\quad A(x, A^{-1}(x,y)) = y. \qquad 2.\quad A({}^{-1}A(x,y), y) = x.$$
$$3.\quad A^{-1}(x, A(x,y)) = y. \qquad 4.\quad {}^{-1}A(A(x,y), y) = x.$$

Here the operation A will be called the **principal** one, and A^{-1} and ${}^{-1}A$, respectively, the **right** and **left inverses** to A. As follows from 1-4, the operations A^{-1} and ${}^{-1}A$ are defined in the following way through the principal operation:

$$A^{-1}(a,b) = c \Longleftrightarrow A(a,c) = b, \qquad {}^{-1}A(a,b) = c \Longleftrightarrow A(c,b) = a.$$

In a quasigroup L, along with the identities 1-4, some other identities may also hold. We shall call the identities 1-4 **basic**, and the remaining ones **additional**. The set of all quasigroups with an additional system of identities Σ will be denoted by $L(\Sigma)$. In particular, if Σ contains the identity $A^{-1}(x,x) = {}^{-1}A(y,y)$, then the quasigroups from $L(\Sigma)$ are called **loops**. If L is a loop and $a \in L$, then the element $A^{-1}(a,a)$ is called its **identity** and is denoted by e; the elements $A^{-1}(a,e)$ and ${}^{-1}A(e,a)$ are called, respectively, the **right** and **left inverses** for a and are denoted by a^{-1} and ${}^{-1}a$. Of the classes of quasigroups that are loops, we shall consider only those in which the identity

$$A^{-1}(x, A^{-1}(x,x)) = {}^{-1}A(A^{-1}(x,x), x),$$

holds; the presence of this identity in the quasigroup L entails $a^{-1} = {}^{-1}a$ for any $a \in L$.

Just as for universal algebras, the notion of a partial quasigroup is defined. In defining a partial loop I, we shall require that I contain an element e and, together with an element a, the element a^{-1}. The set of all partial quasigroups with an additional system of identities Σ will be denoted by $I(\Sigma)$. Each quasigroup of the class $L(\Sigma)$ can be given by generators and defining relations, and therefore for the class $L'(\Sigma)$ of finitely defined quasigroups from $L(\Sigma)$ the basic algorithmic problems are naturally formulated: word identity, isomorphism, and embedding. In the work of T. Evans ([1]) it is proved that if every finite partial algebra of a primitive class $\mathfrak{A}$ is embeddable in an algebra from $\mathfrak{A}$, then for the (finitely defined) algebras of the class $\mathfrak{A}$ the word identity problem is positively solvable. Further, by reducing words to canonical form, the same author established an embedding theorem for some classes of multiplicative systems (and, in particular, for quasigroups and loops) ([2]), which made it possible to solve for these classes of algebras not only the identity problem but also the isomorphism problem ([3]). In this connection we considered most of the classes of quasigroups known from the literature with one additional identity ([4,5]) and with certain systems of identities; for each class

either an embedding theorem for a partial quasigroup in a quasigroup was proved, or a contradictory example was constructed. Examples of classes of quasigroups in which an embedding theorem holds are the classes of quasigroups with each of the following identities (systems of identities):*

$$xy = yx, \quad x(xy) = y, \quad (xy)y = x, \quad x(yx) = y, \quad (xy)x = y, \quad xx = yy,$$

$$T_n = x, \quad n = 1, 2, \dots, \qquad \text{where } T_1 = xx, \quad T_{k+1} = T_k \cdot T_k;$$

$$\begin{cases} x(xy) = y, \\ xy = yx \end{cases} \quad (TS\text{-quasigroups}); \qquad \begin{cases} x(xy) = y, \\ xy = yx, \\ xx = x \end{cases} \quad (\text{Steiner quasigroups}).$$

The embedding theorem also holds for classes of loops with any subsystem of the system of identities (under the condition $x^{-1} = {}^{-1}x$).

$$(xy)x^{-1} = y \quad (CI\text{-loops}); \qquad (xy)^{-1}x = y^{-1} \quad (WIP\text{-loops});$$

$$(xy)y^{-1} = x \quad (RIP\text{-loops}); \qquad x^{-1}(xy) = y \quad (LIP\text{-loops}).$$

$$(xy)^{-1} = y^{-1}x^{-1} \quad (A^*I\text{-loops});$$

To establish the embedding theorem we applied a new method based on the following assertion.

Theorem 1**. *For any class of partial quasigroups the following assertions are equivalent:*

1. *Every partial quasigroup is embeddable in a quasigroup.*
2. *Every finite partial quasigroup is embeddable in a quasigroup.*
3. *Every finite partial quasigroup I is embeddable in its simple finite free extension (i.e. in the quasigroup I_1 obtained from I by adding one new symbol c with a relation of the form $B(a,b)=c$, where B is one of the operations $A, A^{-1}, {}^{-1}A$, $a,b \in I$, and $B(a,b)$ is not defined in I).*

Since I_1 contains no more than one, and in the case of loops no more than two, elements distinct from the elements of I_1, the method that checks the embedding (or nonembedding) of I in I_1 is sufficiently simple. It was established that for all the indicated identities and systems of identities the following condition R is satisfied.

If I_1 is a simple finite free extension of a partial quasigroup I from $I(\Sigma)$, obtained by adding a new element c with the relation $B_1(a,b)=c$, then the new relations appearing thereby: 1) do not depend on the relations in I (but depend only on Σ); 2) contain no elements distinct from $a^{\varepsilon_1}, b^{\varepsilon_2}, c^{\varepsilon_3}$; 3) are distinct from relations of the form $B_2(a^{\varepsilon_1}, a^{\varepsilon_2}) = c^{\varepsilon_3}$ and $B_2(b^{\varepsilon_1}, b^{\varepsilon_2}) = c^{\varepsilon_3}$, if $a \neq b^{\varepsilon}$. (Here $\varepsilon, \varepsilon_i = \pm 1$, and B_i is any of the operations $A, A^{-1}, {}^{-1}A$.)

This fact makes it possible to solve each of the main algorithmic problems and to prove the rank theorem at once for all classes of quasigroups in which the embedding theorem has been established. To this end we proved the following assertions:

Theorem 2. *If for a class of quasigroups $L(\Sigma)$ condition R holds, then in the class $L'(\Sigma)$ the algorithmic problems of word identity, isomorphism, and occurrence are positively decidable.*

Theorem 3. *If for a class of quasigroups $L(\Sigma)$ condition R holds, then the rank (minimal number of generators) of any partial quasigroup I from $I(\Sigma)$ is equal to the rank of any of its simple finite free extensions.*

From Theorem 3, in particular, it follows that for quasigroups of the class $L(\Sigma)$ with condition R the theorem on the minimal number of generators holds; i.e., if a finitely generated quasigroup L from $L(\Sigma)$ is decomposed into a free product of subquasigroups, then the minimal number of generators of L

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* In writing identities, for brevity we write $(\cdot)$ instead of the operation A.

** We note that an analogous theorem can be proved for any primitive class of universal algebras.

is equal to the sum of the minimal numbers of generators of the factors of this decomposition.

In those cases when, in the class of quasigroups $L(\Sigma)$, the embedding theorem does not hold, algorithmic problems may have both a positive and a negative solution. Numerous examples of such classes can be obtained if one uses the passage to parastrophes.

It is known that with each quasigroup L from $L(\Sigma)$ one can associate six quasigroups, taking as the principal operation one of the operations $A, A^{-1}, {}^{-1}A, ({}^{-1}A)^{-1}, {}^{-1}(A^{-1})$, and

$$A^* = [{}^{-1}(A^{-1})]^{-1}$$

([5]). Any two of these quasigroups are called **parastrophic**. Taking, for each quasigroup from $L(\Sigma)$, a parastrophe with principal operation A^σ, we obtain, generally speaking, a new variety of quasigroups $L(\Sigma^\sigma)$, which we shall call **parastrophic** to $L(\Sigma)$.

Theorem 4. *If in some class of quasigroups $L'(\Sigma)$ the word-identity problem, or the embedding problem, or the isomorphism problem is positively decidable, then the corresponding problem is positively decidable in any class of quasigroups $L'(\Sigma^\sigma)$ parastrophic to $L'(\Sigma)$.*

From this theorem, in particular, it follows that the basic algorithmic problems are solved positively for classes of quasigroups parastrophic to the class of all abelian groups, and negatively for classes of quasigroups parastrophic to the class of all groups.

As shown in ([5]), the class of quasigroups with any one of the identities

$$(xy)z = y(zx), \qquad (xy)z = (yz)x, \qquad (xy)z = x(zy)$$

coincides with the class of all abelian groups, and, consequently, in these classes of quasigroups, as well as in the classes parastrophic to them, the problems of word identity, embedding, and isomorphism are positively solved.

Let us note that a parastrophe of an abelian group, generally speaking, is not an abelian group. An example is the class of quasigroups with Neumann' s identity

$$x(y(z(xy))) = z,$$

which is parastrophic to the identity

$$(xy)z = y(zx).$$

On the other hand, the classes of quasigroups with any one of the identities

$$(xy)z = x(yz), \qquad ((xy)z)u = x(y(zu)), \qquad (x(yz))u = (xy)(zu) \tag{1}$$

coincide with the class of all groups, and therefore, on the basis of the results of P. S. Novikov ([6]) and S. I. Adian ([7]), in them the problems of word identity, embedding, and isomorphism are negatively solved.

Hence, and from Theorem 4, there follows a negative solution of the indicated algorithmic problems for classes of quasigroups parastrophic to (1), which, generally speaking, do not coincide with the class of groups.

An example of such a class is furnished by quasigroups with the transitivity identity

$$(xy)(zy) = xz,$$

which is parastrophic to associativity.

There exist many other identities Σ such that the classes of quasigroups $L(\Sigma)$ coincide with the class of all groups. For example, 16 such identities, distinct from (1), are indicated in ([5]).

Let us note that parastrophic quasigroups have the same systems of generators. Hence, from the group-theoretic theorem of Grushko it follows that, for all classes of quasigroups parastrophic to the class of groups, the theorem on the minimal number of generators of a free product of quasigroups holds.

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Note: Figure translations are in progress. See original paper for figures.