

ON THE THEORY OF EXTREMA OF A MULTIDIMENSIONAL ζ -FUNCTION

MATHEMATICS

1967

SovietRxiv

View the original and related papers at <https://sovietrxiv.org/items/ru-196701.52640>

Source: Math-Net.Ru and CyberLeninka. Machine translation. Verify with the original.

Abstract

Full Text

UDC 513.82+511

MATHEMATICS

Corresponding Member of the Academy of Sciences of the USSR B. N. DELONE,
S. S. RYSHKOV

ON THE THEORY OF EXTREMA OF A MULTIDIMENSIONAL ζ -FUNCTION

§ 1. Let an n -dimensional lattice Γ be metrically given, and let $m > n/2$ be a positive number. Then the (n -dimensional) ζ -function of the lattice Γ and of the parameter m is called the infinite series, convergent for $m > n/2$,

$$\zeta(m, \Gamma) = \zeta(m, f) = \sum \frac{1}{r^{2m}} = \sum_{\Gamma} \frac{1}{f^m}, \quad (1)$$

where by r are denoted the distances from some point $O \in \Gamma$ to all the other points of the lattice Γ , and by $f = \sum a_{ij}x_i x_j$ ($i, j = 1, 2, \dots, n$) is denoted the metric form of some fundamental parallelohedron of the lattice Γ . The symbol Γ under the second summation sign means that the sum is taken over all integer systems (x_1, x_2, \dots, x_n) of the variables of the form f , except the system $(0, 0, \dots, 0)$. In what follows we shall omit this symbol.

From the recent investigations of S. L. Sobolev on the theory of approximate computation of multidimensional integrals there follows the great importance, for this theory, of finding such an n -dimensional lattice which would give a minimum of the n -dimensional ζ -function $\zeta(m, \Gamma)$ for a given value of the parameter m and for a fixed determinant Δ of the lattice Γ .

On the way to solving this problem the authors have also constructed the proposed theory of ζ -strongly critical and ζ -finally extremal lattices.

§ 2. **Basic definitions.** We solve the problem indicated above, like any problem of a conditional extremum, by the method of undetermined Lagrange multipliers. Taking the partial derivatives of the Lagrange function $\zeta - \lambda(\Delta - \Delta_0)$ with respect to all a_{ij} , and also with respect to λ , and equating them to zero, we obtain the criticality conditions of the lattice Γ in our problem

$$\sum \frac{x_1^2}{f^{m+1}} = \lambda A_{11}, \quad \sum \frac{x_2^2}{f^{m+1}} = \lambda A_{22}, \dots, \quad \sum \frac{x_{n-1}x_n}{f^{m+1}} = \lambda A_{n-1,n}, \quad \lambda = \frac{\zeta(m, \Gamma)}{n\Delta^2}$$

here by A_{ij} are denoted the algebraic cofactors of the elements a_{ij} in the determinant of the form f , i.e. the coefficients of the reciprocal form. From this system of equations we obtain the proportion

$$A_{11} : A_{22} : \dots : A_{n-1,n} = \sum \frac{x_1^2}{f^{m+1}} : \sum \frac{x_2^2}{f^{m+1}} : \dots : \sum \frac{x_{n-1}x_n}{f^{m+1}}. \quad (2)$$

Definition 1. A lattice Γ (and the corresponding quadratic form f) is called **strongly critical** if it is critical, i.e. satisfies conditions (2), for some infinitely increasing sequence $\{m\}$ of values of the parameter m .

Theorem 1. *In order that a lattice Γ be strongly critical, it is necessary and sufficient that for every k the conditions*

$$A_{11} : A_{22} : \dots : A_{n-1,n} = \sum_k x_1^2 : \sum_k x_2^2 : \dots : \sum_k x_{n-1}x_n, \quad (3)$$

be fulfilled; here by \sum_1 is denoted the sum over the points of the lattice Γ nearest to the point O (i.e. over the minima of the form f), by \sum_2 the sum over the points of the lattice Γ lying at the second greatest distance from the point O , and so on.

Corollary. *A strongly critical lattice is critical for all values of the parameter m greater than $n/2$.*

The proof of the necessity of the first of conditions (3) consists in shortening the right-hand side of proportion (2) by the quantity $1/r_1^{2m}$, where r_1 denotes the minimum of the form f , and then passing to the limit along the sequence $\{m\}$. The necessity of the subsequent conditions is proved similarly, but now using the preceding ones. The corollary is obvious. Let us finally note that the absolute convergence of all the series considered here follows easily from the convergence of series (1).

Definition 2. A lattice Γ is called **finally extremal** if it is ζ -extremal for every m greater than some m_0 .

§ 3. Geometry of the parameter space

Consider three Euclidean spaces: two mutually conjugate n -dimensional spaces Q^n and X^n (the space of variables of the form), and also an N -dimensional one, where $N = n(n+1)/2$, the space V^N (the space of parameters of the form). In each of these spaces its own system of rectangular coordinates $\{q_i\}$, $\{x_i\}$, and $\{v_{ij}\}$, respectively, is given.

Every quadratic form $f = \sum a_{ij}x_ix_j$ will be represented in the space V^N either by the point with coordinates $\{a_{ij}\}$, or by the plane $\sum a_{ij}v_{ij} = n$. The totality of those points $\{v_{ij}\}$ which correspond to positive forms forms an open cone K

with vertex at the origin; the boundary of the cone K —the conical surface \overline{K} —is the totality of points $\{v_{ij}\}$ corresponding to degenerate nonnegative forms.

It turns out that a nonempty finite set is intersected by the cone precisely by those planes which correspond to positive forms. Therefore such planes are called **elliptic**.

Lemma 1. The centers of gravity of all sections of the surface \overline{K} (and of the cone K) by planes parallel to any given elliptic plane $\sum a_{ij}v_{ij} = n$ lie on the straight line containing the points $\{0\}$ and $\{A_{ij}\}$.

Definition 3. The straight line $v_{ij}/A_{ij} = t$ is called the **diameter** conjugate to the plane $\sum a_{ij}v_{ij} = n$.

To finish the paragraph, let us note that the group $\{s\}$ of all unimodular transformations (the group $\{g\}$ of integral unimodular transformations) of the space X^n naturally generates a group $\{S\}$ (the group $\{G\} \subset \{S\}$) of certain transformations of the space V^N .

Lemma 2. The group $\{S\}$ consists of unimodular transformations of the space V^N which transform the surface \overline{K} and the cone K into themselves.

§ 4. The Voronoi mapping and the Voronoi polyhedron

Definition 4. The mapping $v : Q^n \rightarrow V^N$, given by the formula $v_{ij} = q_{iq}j$, is called the **Voronoi mapping**. The image $\{q_{iq}j\}$ of any point $\{q_i\}$ of the space Q^n under the Voronoi mapping is called a **Voronoi point** (of the space V^N); the numbers $\{q_i\}$ are called the **lower coordinates of the Voronoi point**. If all the lower coordinates of a Voronoi point are integers, then it is called an **integral Voronoi point**; if, in addition, they have no common divisor, then the point is called a **primitive Voronoi point**.

All Voronoi points, obviously, lie on the surface \overline{K} of the cone K .

Lemma 3 (on rank). Let an arbitrary figure $\Phi \subset Q^n$ have rank p , i.e. lie in some p -dimensional linear subspace of the space Q^n and not lie in a subspace of smaller dimension. Then the rank of the figure $v(\Phi) \subset V^N$ is not greater than $p(p+1)/2$.

Let us note that the rank of the set of all primitive Voronoi points is equal to N .

Definition 5. The convex hull Π of the set of all primitive Voronoi points is called the **Voronoi polyhedron**.

It is proved^(1,2) that the planes of the $(N-1)$ -dimensional faces of the Voronoi polyhedron are elliptic and, consequently, all its faces are finite. It is also proved there that the number of faces of the polyhedron Π not equivalent with respect to-

with respect to the group $\{G\}$, of course. (Here and in Theorem 2, faces of the polyhedron Π lying entirely in the surface \bar{K} are not considered.)

Let us note that Lemma 3 greatly facilitates the usual proofs.

§ 5. Geometric interpretation of criticality and strong criticality. Finiteness of the number of strongly critical lattices. Geometrically, in the parameter space, the criticality of a lattice Γ , i.e., the fulfillment for it of condition (2), means that the center of gravity of all Voronoi integer points, taken with weights equal to $1/r^{2m}$ (see § 1), lies on the diameter conjugate to the plane corresponding to the form f .

The strong criticality of a lattice Γ , i.e., the fulfillment for it of conditions (3), means geometrically that: 1) the center of gravity of the vertices of the polyhedron Π lying on its supporting plane P_1 , given by the equation $\sum a_{ij}v_iv_j = \text{const}$, i.e., on the plane corresponding to the first distance r_1 of the lattice Γ , lies on the diameter conjugate to this plane; 2) the center of gravity of the Voronoi points lying in the plane P_2 , parallel to the plane P_1 and corresponding to the second distance of the lattice Γ , also lies on the diameter conjugate to the plane P_1 , etc.

Theorem 2. *There exists only a finite number of n -dimensional strongly critical lattices.*

Indeed, the simplest necessary condition for the strong criticality of a lattice Γ is the fulfillment for it of the first of the conditions listed. Consequently, in order to find a lattice that may be strongly critical, one must take a diameter d containing the center of gravity of an arbitrary face of the polyhedron Π (of any dimension, provided only that this face does not belong entirely to the surface \bar{K}) and construct that supporting plane P of the polyhedron Π to which the diameter d is conjugate. If the plane P contains the face from which it was constructed, then the quadratic form corresponding to this plane may turn out to be strongly critical. If it does not contain it, then the face under consideration does not correspond to a strongly critical lattice. Thus, to each face of the polyhedron Π there corresponds no more than one strongly critical form, i.e., to each family of equivalent faces there corresponds no more than one strongly critical lattice, and there are only finitely many such families.

Thus we have proved that it is possible, in a finite number of operations, to compute all lattices (in finite number) that could be strongly critical. However, we do not yet know how completely to avoid the circumstance that checking such a form for strong criticality requires verifying an infinite number of conditions (3).

§ 6. One sufficient condition for the strong criticality of a lattice. Suppose that some subgroup $\{S_P\}$ of the group $\{S\}$ maps a certain ellipsoid plane P into itself; then it also maps into themselves all planes parallel to this one. Suppose, further, that the group $\{S_P\}$ is such that in the plane P there is only one absolutely fixed point of this group. Since the diameter conjugate

to the plane P is also mapped into itself, this sole absolutely fixed point is the point of its intersection with the plane P . If a certain group $\{S_P\} \subset \{S\}$ has these properties, then we shall call it **strong**.

The following important lemma holds:

Lemma 4. *In order that the group $\{S_P\}$ be strong, it is necessary and sufficient that the group $\{s_P\}$ that generated it have no absolutely fixed subspaces, i.e., subspaces mapped into themselves under every transformation from the group $\{s_P\}$.*

Theorem 3. *In order that a form f be strongly critical, it is sufficient that the group $\{G\}$ have a strong subgroup $\{G_P\}$ of transformations of the plane P , corresponding to the form f , into itself.*

For the proof, take some plane P_q , parallel to the plane P , on which lie Voronoi points corresponding to the points

lattice lying at distance r_q from the point O . Since this plane P_q , under all transformations from the group $\{G_P\}$, is carried into itself, and all Voronoi points under these transformations are carried into Voronoi points, the center of gravity of the Voronoi points lying in this plane is carried into itself under all transformations from the group $\{G_P\}$. Consequently, this center of gravity must lie on the diameter conjugate to the plane P , i.e., the conditions of strong criticality of the form f are satisfied.

Since the group $\{g_P\}$, which generates the group $\{G_P\}$, is nothing other than the group of rotations (or a subgroup of it) of the lattice Γ into itself, described in its frame corresponding to the form f , Theorem 3, by virtue of Lemma 4, can be formulated as follows:

Theorem 3'. *In order that the lattice Γ be strongly critical, it is sufficient that the group of its rotations have no absolutely fixed subspaces.*

§ 7. **Final extremality of lattices.** The following theorem is proved by a detailed consideration of the quadratic terms of the Taylor expansions of the determinant of the form and of the function $\zeta(m, f)$, as well as by considering the geometry of the Voronoi polyhedron.

Theorem 4. *A quadratic form f is finally extremal if and only if it is perfect, i.e., corresponds to an $(N - 1)$ -dimensional face of the Voronoi polyhedron, and is strongly critical.*

Corollary. *Every n -dimensional finally extremal lattice is extreme, i.e., corresponds to a locally densest arrangement of n -dimensional spheres.*

§ 8. **Examples.** In all dimensions, an example of a strongly critical but not finally extremal lattice may be the cubic lattice, i.e., the lattice constructed on the frame with metric form $x_1^2 + x_2^2 + \dots + x_n^2$. In all dimensions, an example of a finally extremal lattice may be the lattice constructed on the regular simplex, i.e., on the frame with metric form $\varphi_0^n = x_1^2 + x_2^2 + \dots + x_n^2 + x_1x_2 + \dots + x_{n-1}x_n$.

In two-dimensional space there exist only the two strongly critical lattices described above, and the lattice constructed on the regular triangle is optimal in our problem ⁽³⁾.

In three-, four-, and five-dimensional spaces there are respectively one, two, and three perfect lattices; all of them are finally extremal. Moreover, the form φ_0^3 is extremal for every integer m ⁽⁴⁾.

In six-dimensional space there is a perfect form $\varphi_5^6 = 2\varphi_0^6 - x_1x_2 - x_4x_5 - x_4x_6 - x_5x_6$, which is not even extreme ⁽⁵⁾; there is the perfect extreme, but not finally extremal, form $\varphi_3^6 = 2\varphi_0^6 - x_1x_2 - x_3x_4 - x_5x_6$; the remaining five ⁽⁶⁾ six-dimensional perfect forms are finally extremal.

Finally, let us note that the initiative in seeking finally extremal and strongly critical lattices belongs to S. S. Ryshkov.

Steklov Mathematical Institute
Academy of Sciences of the USSR

Received
2 I 1967

CITED LITERATURE

- ¹ G. F. Voronoi, *Collected Works*, 2, Kiev, 1952, pp. 171-214.
- ² B. A. Venkov, in the book G. F. Voronoi, *Collected Works*, 2, Kiev, 1952, pp. 379-385.
- ³ B. N. Delone, N. N. Sandakova, S. S. Ryshkov, DAN, **162**, No. 6 (1965).
- ⁴ N. N. Sandakova, DAN (1967) (in press).
- ⁵ V. S. Vladimirova, *Matem. sbornik*, **44**, 86 (1958).
- ⁶ E. S. Barnes, *Phil. Trans. Roy. Soc. London*, A **249**, 461 (1957).

Note: Figure translations are in progress. See original paper for figures.

Source: Math-Net.Ru and CyberLeninka. Machine translation. Verify with the original.