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Abstract

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MATHEMATICS

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ON THE MULTIPLICITIES OF WEIGHTS OF REPRESENTATIONS

AND THE MULTIPLICITIES OF REPRESENTATIONS OF SEMISIMPLE LIE ALGEBRAS

(Presented by Academician I. M. Vinogradov, 28 IX 1966)

Finite-dimensional representations of semisimple complex Lie algebras will be considered. A recurrence formula is given for the multiplicities of the weights of an irreducible representation. Also considered are the problems of decomposing the tensor product of irreducible representations into irreducible representations and of decomposing an irreducible representation of a semisimple Lie algebra into irreducible representations of its semisimple subalgebra.

Let G be a complex semisimple Lie algebra of rank l, and let H be its Cartan subalgebra. The roots of the algebra and the weights of its representations may be regarded as linear forms on H. To each simple root a_i $(i = 1, \dots, l)$ we associate a vector $h_{a_i} \in H$ satisfying the condition

$$a_i(h) = (h_{a_i}, h)$$

for all $h \in H$, where (h_{a_i}, h) is the scalar product of the vectors h_{a_i} and h, determined by the Killing-Cartan form. The set of vectors $h_i = 2h_{a_i}/(a_i, a_i)$, $i = 1, \dots, l$, is a basis of the space H. The set K of all linear forms Λ whose values on the basis vectors $\Lambda(h_i)$ are integers forms the set of all weights of all finite-dimensional representations of the algebra G. A weight $\Lambda \in K$ is called **dominant** if $\Lambda(h_i) \geq 0$, $i = 1, \dots, l$. Elements of K obtained from a given element under the action of transformations from the Weyl group W of the algebra G are called **equivalent**. The dominant element equivalent to a given element $M \in K$ will be denoted by $\{M\}$. To each element $M \in K$ we assign a number β_M, equal to 0 or ± 1, with $\beta_M = 0$ if there exists a nonidentity element $S \in W$ such that $SM = M$, and $\beta_M = \det T$, $T \in W$, if no such S exists and $TM = \{M\}$.

If $D_{\Lambda'}$ and $D_{\Lambda''}$ are irreducible representations of the algebra G with highest weights Λ' and Λ'', respectively, then, in view of complete reducibility of representations,

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$$D_{\Lambda'} \otimes D_{\Lambda''} = \sum_{\Lambda_i} \oplus m_{\Lambda_i} D_{\Lambda_i}, \tag{1}$$

where m_{Λ_i} is the multiplicity of the irreducible representation D_{Λ_i} in the decomposition of the representation $D_{\Lambda'} \otimes D_{\Lambda''}$ into irreducible representations, and the summation is over all dominant weights in K. R. Brauer and G. Weyl (^{1}) derived a formula expressing m_{Λ_i} through the multiplicities of the weights of one of the multiplied representations. Their formula may be written in the form

$$m_{\Lambda_i} = \sum_{S \in W} \det S \, n_{\Lambda_i + R - S\Lambda'' - SR}, \tag{2}$$

where R is the half-sum of the positive roots of the algebra G, $n_{\Lambda_i + R - S\Lambda'' - SR}$ is the multiplicity of the weight $\Lambda_i + R - S\Lambda'' - SR$ in the representation $D_{\Lambda'}$, and the summation is over all elements of the Weyl group W of the algebra G.

Using formula (2), the following theorem is proved.

Theorem 1. If D_Λ is an irreducible representation of the complex semisimple Lie algebra G with highest weight Λ, then for the multiplicities of the weights of this representation the following relation holds*

$$\sum_{S \in W} \det S \, n_{\Lambda_i + R - SR} = \det T, \quad \text{if } \{\Lambda_i + R\} = \Lambda + R, \tag{3}$$

where $T \in W$ is determined from the condition $T(\Lambda + R) = \Lambda_i + R$, or

$$\sum_{S \in W} \det S \, n_{\Lambda_i + R - SR} = 0, \quad \text{if } \{\Lambda_i + R\} \neq \Lambda + R. \tag{4}$$

From Theorem 1 there easily follows a theorem giving a recurrent formula for the multiplicities of weights of a representation:

Theorem 2. If D_Λ is an irreducible representation of the complex semisimple Lie algebra G with highest weight Λ, Λ_i is a weight of this representation with nonzero multiplicity, and $\Lambda_i \neq \Lambda$, then

$$n_{\Lambda_i} = - \sum_{\substack{S \in W \\ S \neq 1}} \det S \, n_{\Lambda_i + R - SR}. \tag{5}$$

It is known that if S is a nonidentity element of W, then $R - SR$ is a nonzero sum of distinct positive roots. Therefore, since every weight of the representation has the form

$$\Lambda - \sum_{i=1}^{l} k_i \alpha_i,$$

where Λ is the highest weight, k_i are nonnegative integers, and α_i are simple roots, formula (5) expresses the multiplicity of the weight

$$\Lambda - \sum_{i=1}^{l} k_i \alpha_i$$

in terms of the multiplicities of certain weights of the form

$$\Lambda - \sum_{i=1}^{l} k_i' \alpha_i,$$

where $0 \le k_i' \le k_i$ and

$$\sum_{i=1}^{l} k_i' \alpha_i \ne \sum_{i=1}^{l} k_i \alpha_i.$$

Formula (5) is more effective than Kostant’ s formula (2) for weight multiplicities or Freudenthal’ s recurrent formula (3).

Lemma 1. For the multiplicities of weights of an irreducible representation of the algebra G, the following relation holds

$$\sum_{S\in W} \det S\, n_{\Lambda_i - S\tilde{\Lambda}} = \det T \sum_{S\in W} \det S\, n_{\{\Lambda_i\} - S\tilde{\Lambda}}, \quad \Lambda_i \in K, \quad \tilde{\Lambda} \in K, \tag{6}$$

where $T \in W$ sends Λ_i to $\{\Lambda_i\}$. If there exists $T_1 \in W$ such that $T_1 \Lambda_i = \Lambda_i$, then

$$\sum_{S\in W} \det S\, n_{\Lambda_i - S\tilde{\Lambda}} = 0. \tag{7}$$

With the aid of formula (2), Lemma 1, and Theorem 1, the following theorem is easily proved.

Theorem 3. For the multiplicities m_Λ of irreducible representations D_Λ in the direct product of irreducible representations $D_{\Lambda'}$ and $D_{\Lambda''}$ of the algebra G, the following relation holds

$$\sum_{T\in W} \det T\, \beta_{\Lambda_i + 2R - TR}\, m_{\{\Lambda_i + 2R - TR\} - R} = \sum_{S\in W} \gamma_{\Lambda_i + R - S\Lambda'' - SR} \det S, \tag{8}$$

where $\gamma_M = 0$ if there does not exist $T_1 \in W$ such that $T_1(M+R) = \Lambda' + R$, and $\gamma_M = \det T_1$ if $T_1(M+R) = \Lambda' + R$.

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[*] If M, $M \in K$, is not a weight of the representation, then we assume that $n_M = 0$.

Restrict the irreducible representation D_Λ of the algebra G to its semisimple subalgebra G', i.e. consider D_Λ only on elements of G'. In view of complete reducibility, the representation D_Λ of the subalgebra G' decomposes into a direct sum of irreducible representations

$$D_\Lambda = \sum_{\Lambda_i} \oplus\, m_\Lambda(\lambda_i) D'_{\lambda_i}, \tag{9}$$

where D'_{λ_i} is an irreducible representation of the subalgebra G' with highest weight λ_i; $m_\Lambda(\lambda_i)$ is the multiplicity of the irreducible representation D'_{λ_i} in the representation D_Λ, and the sum is taken over all dominant weights from the set K' of all weights of all finite-dimensional representations of the subalgebra G'.

We shall agree to denote by $\overline{M}$ the weights $M \in K$ restricted to the Cartan subalgebra H' of the subalgebra G', i.e. considered as linear forms only on H'.

Lemma 2. Let D_Λ be an irreducible representation of the algebra G; let G' be its semisimple subalgebra satisfying the following condition. If in G' a Cartan subalgebra H' has been chosen that is contained in the Cartan subalgebra H of the algebra G, then the weight diagrams* of irreducible representations of the algebra G, considered only on H', are invariant with respect to transformations from the Weyl group W' of the subalgebra G'. Then for the multiplicities of the weights of the representation D_Λ the relation

$$\sum_{S' \in W'} \det S' \sum_{\overline{\Lambda} j = \nu_1 - S'\nu_2} n_{\Lambda_j} = \det T' \sum_{S' \in W'} \det S' \sum_{\overline{\Lambda} j = \{\nu_1\} - S'\nu_2} n_{\Lambda_j}, \tag{10}$$

$$\nu_1 \in K', \qquad \nu_2 \in K',$$

holds, where $T' \in W'$ is determined from the condition $T'\nu_1 = \{\nu_1\}$, and the second sums on the right- and left-hand sides of (10) are taken over those weights Λ_j of the representation D_Λ for which $\overline{\Lambda}_j$ is equal to $\{\nu_1\} - S'\nu_2$ $(\nu_1 - S'\nu_2)$. If there exists $T'_1 \in W'$ for which $T'_1\nu_1 = \nu_1$, then

$$\sum_{S' \in W'} \det S' \sum_{\overline{\Lambda} j = \nu_1 - S'\nu_2} n_{\Lambda_j} = 0. \tag{11}$$

With the aid of this lemma and Theorem 1 the following theorem is proved.

Theorem 4. If G is a complex semisimple algebra, G' is its semisimple subalgebra satisfying the condition of Lemma 2, then for the multiplicities $m_\Lambda(\lambda_i)$ of the irreducible representations D'_{λ_i} of the subalgebra G' in the irreducible representation D_Λ of the algebra G the relation

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$$\sum_{S \in W} \det S \, \beta_{\lambda_i + \overline{(R - SR)} + R'} \, m_\Lambda(\{\lambda_i + \overline{(R - SR)} + R'\} - R') =$$

$$= \sum_{S' \in W'} \det S' \, \delta_{\overline{\Lambda}, \, \lambda_i + R' - S' R'}, \tag{12}$$

holds, where $\delta_{\overline{\Lambda}, \, \lambda_i + R' - S' R'}$ is the Kronecker symbol.

Let $M \in K$ and

$$F_M(h) = \sum_{S \in W} \det S \, \exp(SM, h), \qquad h \in H. \tag{13}$$

Then, if $T_1, T_2, \dots, T_n$ is the set of all transformations from W for which $T_{iM} = M$, then

$$\sum_{i=1}^{n} \det T_i \, \exp(T_{iM}, h) = 0. \tag{14}$$

Using formulas (2), (13), and (14), it is easy to prove Theorem 5.

Theorem 5. *If $D_{\Lambda'}$ and $D_{\Lambda''}$ are irreducible representations of the algebra G, then*

$$D_{\Lambda'} \otimes D_{\Lambda''} = \sum_{\Lambda'_j} n_{\Lambda'_j} \, \beta_{\Lambda'_j + \Lambda'' + R} \, D_{\{\Lambda'_j + \Lambda'' + R\} - R}, \tag{15}$$

where the summation is over all weights of the representation $D_{\Lambda'}$, $n_{\Lambda'_j}$ is the multiplicity of the weight Λ'_j in $D_{\Lambda'}$, and the sum obtained after collecting like terms will be direct.

A similar theorem holds for the case of restricting a representation of the algebra to a subalgebra:

Theorem 6. *If D_Λ is an irreducible representation of the semisimple algebra G, and G' is a semisimple subalgebra of the algebra G satisfying the condition of Lemma 2, then for the restriction of the representation D_Λ to G' the formula*

$$D_\Lambda = \sum_{\Lambda_j} n_{\Lambda_j} \, \beta_{\overline{\Lambda} j + R'} \, D_{\{\overline{\Lambda} j + R'\} - R''}, \tag{16}$$

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* By a weight diagram is meant the totality of all weights of a representation together with their multiplicities.

holds, where the summation is over all weights of the representation D_Λ*, and the sum obtained after collecting like terms will be direct.*

We note that regular semisimple subalgebras satisfy the condition of Lemma 2. However, the class of subalgebras satisfying this condition is not exhausted by regular subalgebras.

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Note: Figure translations are in progress. See original paper for figures.

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