TOPOLOGICAL PROPERTIES

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Abstract

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OF PARTIALLY ORDERED DYNAMICAL SYSTEMS

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1°. The theory of dynamical systems, begun by Birkhoff, has undergone further development and generalization in many works ([1,2]). In particular, E. A. Barbashin ([3]) introduced partially ordered dynamical systems into consideration. In contrast to ordinary ([4]) and general ([5,2]) dynamical systems, here, instead of the group of real numbers or another topological group, an arbitrary partially ordered group is considered. The partial ordering of the phase group used in ([3]) differs from the ordinary one ([6]) by the existence in the group of an ω-sequence, i.e., a countable increasing sequence of elements majorizing every element of the group. In the present note this restriction is removed and, only in some cases, replaced by the more natural, from the algebraic point of view, condition that the group be directed. As is clear from ([7]), this makes it possible to consider a broader class of groups. It is shown below that, under this condition, many of the basic concepts and theorems of the topological theory of dynamical systems can be carried over to partially ordered dynamical systems.

2°. Let G be a nontrivially partially ordered group ([6]), and let e be the identity of the group G,

$$G^+ = \{g \in G: \ g \geq e\}, \qquad (g_1, g_2) = \{g \in G: \ g_1 < g < g_2\}.$$

Let $K \subseteq G$. We shall call the set K **relatively dense** in G (G^+) if there exists an element $g_0 \in G^+ \setminus e$ such that

$$K \cap (g, gg_0) \neq \Lambda,$$

whatever the element $g \in G$ ($g \in G^+$) may be. A set $K \subseteq G$ will be called an ω**-set** if for every element $g \in G$ there exists an element $k \in K$ such that $g < k$. Obviously, every set relatively dense in G is an ω-set.

As usual, we shall call the group G **directed** if the following Moore–Smith axiom is satisfied: for any $g_1, g_2 \in G$ there exists an element $g \in G$ such that

$g > g_1$ and $g > g_2$. In the case of a directed group G, every set $K \subseteq G$ relatively dense in G^+ is an ω-set. A set $K \subseteq G$ is called **ω-directed** if for any $k_1, k_2 \in K$ there exists an element $k \in K$ such that $k > k_1$ and $k > k_2$.

3°. By a **partially ordered dynamical system** we shall mean a collection $[R, G, f]$, consisting of a metric space R, a nontrivially partially ordered group G, and a function f mapping $R \times G$ into R and possessing the following properties:

1) $f(p, e) = p$ for every $p \in R$.
2) $f(f(p, g_1), g_2) = f(p, g_1 g_2)$ for all $g_1, g_2 \in G,\ p \in R$.
3) For any point $p \in R$, element $g \in G$, and number $\varepsilon > 0$, there exists a $\delta > 0$ such that, whatever the point $q \in S(p, \delta)$ may be, the inequality holds

$$\rho(f(p, g), f(q, g)) < \varepsilon. \tag{1}$$

Sometimes, as in ([3]), instead of condition 3) a stronger condition (of **integral continuity**) will be used.

Whatever the element $g_0 \in G^+ \setminus e$, the number $\varepsilon > 0$, and the point $p \in R$ may be, there exists a $\delta > 0$ such that (1) is satisfied for all $g \in S(p, \delta)$ and $g \in (e, g_0)$.

Remark. In all propositions of the present paper in which conditions of integral continuity or directedness of the group occur, the necessity of these conditions is proved by constructing the corresponding examples.

Let $A \subseteq R,\ K \subseteq G$. Denote

$$f(A, K) \equiv \bigcup_{p \in A,\ g \in K} f(p, g), \quad \Sigma_A \equiv \overline{f(A, G)}, \quad \Sigma_A^+ \equiv \overline{f(A, G^+)}.$$

The function $f(p, g)$, for fixed p, is called a **motion**, and the set $f(p, G)$ the **trajectory** of the point p. A set $A \subseteq R$ is called **invariant** if $f(A, g) = A$ for every $g \in G$. An invariant closed set $A \subseteq R$ is called **minimal** if it contains no proper invariant closed subset.

4°. Let $A \subseteq R,\ K \subseteq G$, and let a mapping $\varphi : K \to R$ be given. We shall say that the set A **contains arbitrarily late points** of the mapping φ if for every element $k_0 \in K$ there exists a $k \in K$ such that $k > k_0$ and $k\varphi \in A$. A point $q \in R$ will be called an ω-**limit point** of the mapping φ if every neighborhood $U(q)$ contains arbitrarily late points of this mapping. The set of all ω-limit points of the mapping φ will be denoted by Ω_φ.

Lemma 1. *If the set $K \subseteq G$ is ω-directed, and every neighborhood of the compact set $A \subseteq R$ contains arbitrarily late points of the mapping φ, then $A \cap \Omega_\varphi \neq \Lambda$.*

A point $q \in R$ will be called an **ω-limit point** of the motion $f(p,g)$ if it is an ω-limit point of the mapping $f_p : G \to R$, where $f_p(g) = f(p,g)$. The set of all ω-limit points of the motion $f(p,g)$ will be denoted by Ω_p and called the **ω-limit set** of the motion $f(p,g)$. Obviously this set is closed. It is not difficult to show that Ω_p is invariant, and if $q \in f(p,G)$, then $\Omega_q = \Omega_p$, while if $q \in \Sigma_p^+$, then $\Omega_q \subseteq \Omega_p$. We shall say that the motion $f(p,g)$ is **Lagrange stable in the positive direction** (**Lagrange stable**) if Σ_p^+ (Σ_p) is compact.

On the basis of Lemma 1 one easily obtains

Theorem 1. *If the group G is directed, then the ω-limit set of a motion that is Lagrange stable in the positive direction is nonempty.*

5°. We shall say that $f(p,G^+)$ $(f(p,G))$ **uniformly approximates** the set $Q \subseteq R$ if, for every $\varepsilon > 0$, there exists an element $g_\varepsilon \in G^+ \setminus e$ such that for every pair of points p_1 and p_2, $p_1 \in f(p,G^+)$ $(p_1 \in f(p,G))$, $p_2 \in Q$, one can indicate an element $g \in (e, g_\varepsilon)$ such that $\rho(f(p_1,g),p_2) < \varepsilon$. For ordinary dynamical systems this definition coincides with the definition of uniform approximation of V. V. Nemytskii ([4]). The motion $f(p,g)$ will be called **almost recurrent** (**recurrent**) if its trajectory uniformly approximates the points $p(f(p,G))$.

The following theorems are analogues of the theorems of Birkhoff and Bebout ([8]).

Theorem 2. *For a directed group G, in a compact minimal set all motions are recurrent.*

Theorem 3. *Under the condition of integral continuity, the closure of the trajectory of an almost recurrent motion is a minimal set.*

Corollary 1. *In a compact space, under the condition of integral continuity, the closure of the trajectory of a recurrent motion is a compact minimal set.*

6°. Following (9), we shall say that a point $p \in B \subseteq R$ and a motion $f(p,g)$ are **positively stable in the sense of Lyapunov** with respect to the set B, if for every $\varepsilon > 0$ there exists a $\delta > 0$ such that, for every point $q \in B$ satisfying $\rho(p,q) < \delta$, inequality (1) is satisfied for all $g \in G^+$. We shall say that a set $\Sigma \subseteq R$ has the S^+-property if for every $\varepsilon > 0$ there exists a $\delta > 0$ such that inequality (1) is satisfied whenever $g \in G^+$, $p, q \in \Sigma$, and $\rho(p,q) < \delta$.

Lemma 2. *Let $f(p,G^+)$ have the S^+-property. Then Σ_p^+ has the S^+-property, and Ω_p, if it is nonempty, is a minimal set.*

We shall say that an element $g \in G$ is an **ε-displacement** of the trajectory $f(p,G)$ if $\rho(f(q,g),q) < \varepsilon$ for all $q \in f(p,G)$. A motion $f(p,g)$ will be called **almost periodic** if for every $\varepsilon > 0$ there exists a relatively dense set in G of ε-displacements of the trajectory $f(p,G)$.

The following theorem is an analogue of the corresponding theorem of A. A. Markov ([4]).

Theorem 4. *Let the group G be commutative. If the motion $f(p, g)$ is almost recurrent and positively stable in the sense of Lyapunov with respect to $f(p, G)$, then it is almost periodic.*

From Lemma 2 and Theorems 2 and 4 it follows that

Theorem 5. *Let the group G be directed and commutative. If the motion $f(p, g)$ is stable in the sense of Lagrange in the positive direction, and $f(p, G^+)$ has the S^+-property, then Ω_p is a compact minimal set of almost periodic motions.*

This proposition is a generalization of the corresponding theorem of V. V. Nemytskii ([10]). For ordinary dynamical systems it was first established in ([11]) and later in ([12]).

7°. Analogously to the case of ordinary dynamical systems ([13]), we shall call a point q a ψ-limit point of the motion $f(p, g)$ if $f(p, G^+)$ uniformly approximates the point q. The set of all ψ-limit points of the motion $f(p, g)$ will be denoted by Ψ_p. It is easy to prove that Ψ_p is closed. Obviously, a point q is a ψ-limit point of the motion $f(p, g)$ if and only if for every $\varepsilon > 0$ there exists a relatively dense set $K \subseteq G$ in G^+ such that $f(p, K) \subseteq S(q, \varepsilon)$.

Lemma 3. *If the group G is directed, then $\Psi_p \subseteq \Omega_p \subseteq \Sigma_p^+$, and the set Ψ_p is invariant.*

Lemma 4. *If the group G is directed, the motion $f(p, g)$ is stable in the sense of Lagrange in the positive direction (stable in the sense of Lagrange), and M is the unique minimal set in Ω_p (Σ_p), then $f(p, G^+)$ ($f(p, G)$) uniformly approximates M.*

This lemma is used in the proof of the following theorem.

Theorem 6. *Let integral continuity hold, and let the group G be directed. In order that the positive part of the trajectory $f(p, G^+)$ (the trajectory $f(p, G)$) of a motion $f(p, g)$, stable in the sense of Lagrange in the positive direction (stable in the sense of Lagrange), uniformly approximate some subset $Q \subseteq \Omega_p$ ($Q \subseteq \Sigma_p$), it is necessary and sufficient that Σ_Q be the unique minimal set in Ω_p (Σ_p).*

This theorem for ordinary dynamical systems was proved by K. S. Sibirskii ([11,13]).

Corollary 2. *Under the condition of integral continuity and directedness of the group G, the set Ω_p for a motion $f(p, g)$ stable in the sense of Lagrange in the positive direction is a minimal set of recurrent motions if and only if $f(p, G^+)$ uniformly approximates Ω_p.*

This proposition shows that, under certain conditions, for partially ordered dynamical systems as well there holds a property established for ordinary dynamical systems by V. V. Nemytskii ([4,10]).

8°. A pair (p, q) $(p, q \in R)$ will be called **proximal** ([14]) if for every $\varepsilon > 0$ there exists an element $g \in G$ such that inequality (1) is satisfied. Thus, in R there is defined a reflexive, symmetric, but, generally speaking, nontransitive relation of proximality.

We shall say that a pair (p, q) is **relatively densely proximal** ([15]) if for every $\varepsilon > 0$ there exists a relatively dense set $K \subseteq G$ such that condition (1) is satisfied for all $g \in K$.

A pair (p, q) will be called **uniformly proximal** if $f(p, G)$ and $f(q, G)$ synchronously uniformly approximate some point $r \in R$, i.e., there exists a point $r \in R$ such that for every $\varepsilon > 0$ there is a relatively dense set $K \subseteq G$ such that $f(p, K) \subseteq S(r, \varepsilon)$ and $f(q, K) \subseteq S(r, \varepsilon)$. It is easy to give an example showing that, even in a compact space, the condition of uniform proximality is stronger than relatively dense proximality.

For partially ordered dynamical systems the following lemma, due to Auslander ([16]), remains valid.

Lemma 5. *If the proximality relation in R is transitive, then it is transitive also for the dynamical system induced in $R \times R$.*

With the aid of Lemmas 4 and 5, the following has been proved.

Theorem 7. *If the group G is directed, the space R is compact, and the proximality relation is transitive, then it is a relation of uniform proximality.*

This proposition generalizes Wu' s theorem ([17]) that if the proximality relation is transitive and R is compact, then this relation is a relatively densely proximal relation. It shows that, in compact R, the concept of transitive proximality coincides with the concept of synchronous uniform approximation by the trajectories $f(p, G)$ and $f(q, G)$ of some point in R, or, equivalently, with the concept of uniform approximation by the trajectory $f((p, q), G)$ of some point on the diagonal of $R \times R$.

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Note: Figure translations are in progress. See original paper for figures.