

## On the theory of Painlevé' s third equation

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### Abstract

For the third Painlevé equation, the nature of possible singular points of its solutions is investigated, the question regarding the number and residues of movable poles of the solutions is resolved, and necessary and sufficient conditions for the existence of rational solutions are specified. Bibliography: 5 items.

### Full Text

#### Preamble

This work, originating from 1967, examines the differential equation of the form:

$$zww'' = zw'^2 - ww' + aw^3$$

and its related systems [2]. We consider the parameters  $\alpha, \beta, \gamma, \delta$  and the system (1). By substituting the expressions for  $u$  and  $v$ :

$$\begin{aligned} u &= a_0(z) + a_1(z)w + a_2(z)w^2 + a_3(z)w^3 \\ v &= b_0(z) + b_1(z)w + b_2(z)v + b_3(z)wv + b_4(z)wv^2 \end{aligned}$$

into the governing equations, we derive the relationships between the coefficients  $a_j(z)$  and  $b_j(z)$ . Specifically, we find that  $-a_0a_1 = -(\beta - a_0)$ ,  $b_3 + a_2 = 0$ ,  $a_1 + ba_3 = \gamma$ , and  $b_4 + b_3 = 0$ . Given  $a_3 = \delta$ ,  $b_1 = 0$ , and  $b = 0$  in system (1), where  $b \neq 0$  and  $a_3 \neq 0$ , the condition  $a^2 - \delta a_3 + \beta a_3 + b_2 a_3 = \gamma$  must be satisfied.

Following the methodology of N. A. Lukashevich, we consider the transformation  $u = zw$ , which leads to:

$$\begin{aligned} z \frac{dw}{dz} &= a_0z + hw + w^2u \\ u &= az + \gamma z^2w - ha - wu^2 \end{aligned}$$

For the case where  $b \neq 0$ , system (1) can be rewritten as  $z \frac{dw}{dz} = aw + zw \pm zw$ . The relationship  $(a - 1)ww' + zw^2$  and  $a = -1, w = \phi(z, a, \beta, \gamma, \delta)$  for

system (1). In (1), let  $z' = e^{2t}$ . If  $8w^2 \neq 0$ , then from (7) and (1) we have: a)  $w = \phi^{-1}(z, -\beta, -\alpha, -\delta, -\gamma)$ , b)  $w = \phi^{-1}(-z, \beta, \alpha, -\delta, -\gamma)$ , c)  $w = -\phi^{-1}(z, \beta, \alpha, -\delta, -\gamma)$ , d)  $w = \phi^{-1}(z, -\beta, -\alpha, -\delta, -\gamma)$ , and  $8w' = e^t = 1$ . At  $z = 0$ , for cases a) through d), we observe that  $z = 0$  is a singular point. We note that at  $z = 0$ , the condition  $\alpha = \gamma = 0$  holds. Thus, at  $z = 0$ , we have the expansion  $a_0\phi + a_2z^2$  for (1) where  $z, \alpha, \gamma$  satisfy  $(1 - 3a_{-1} - 4\gamma a_{-1})a_0 = 0$  and  $(4 - 3\alpha a_{-1} - 4\gamma a_{-1})A = 3(9 - 3\alpha a_{-1} - 4\gamma a_{-1})a_{n-1} + \alpha(a_n + ba_{n-1}) + \gamma(4a_{-1}a_0 + 12a_n) + \beta a_0[(n+1)^2 - 3\alpha a_{-1} - 4\gamma a_{-1}]a_{n-1}a_n = P_n(a_{-1}, a_0, a_n)$  for  $n = 0, 1, 2, \dots$ . The coefficients  $a_n$  are given by:

$$[(n+1)^2 - \alpha^2]a_{n-1}P_n(a_{-1}, a_0, \dots, a_{n-1}) = a_n(\alpha, \beta, \delta) \quad (10)$$

From (10), at  $z = 0$  for system (1), if  $k^2\gamma - \alpha^2 \neq 0$ , then  $a_{n-1} = 0$ , and for  $j = 0, 1, 2, \dots$ , we have  $k^2\gamma - \alpha^2 = 0$ . In (8), the coefficients  $a_j$  are determined. Setting  $wu = 1$  in (6), we obtain  $z - \alpha u - zuw_x$  and  $zuw^2$ . Let  $a = r = 2\gamma$  and  $x, \alpha, v$  satisfy (11). Then  $w_1 = J_x y + v = b + v$ . From  $2\sqrt{\gamma} - \alpha z - z(b+v)\alpha^2$ , we get  $z = (a-1)v + zu[b + (b+\alpha)]$  (12). As  $z \rightarrow 0$ , we have  $(u, v) \rightarrow 0$  if  $a_{n-1} = 0$ , meaning  $\alpha$  is not an integer. If  $\alpha$  is an integer, then  $(u, v) \rightarrow 0$  as  $z \rightarrow 0$ . Following N. A. Lukashovich for  $z = 0$  [3], from (8) we have  $a = 2\sqrt{\gamma}$ . If  $\beta \neq 0$  and  $\beta = 0$ , then  $(w, u) \rightarrow 0$  as  $z \rightarrow 0$ . If  $\beta = \delta = 0$ , then  $w \rightarrow 0$  as  $z \rightarrow 0$ . If  $\alpha = \gamma = 0$ , then for system (1) at  $z = 0$ , we have the series (13). In (1) with  $a_0 = 0$  and  $\beta\delta \neq 0$ , the coefficients  $a_j$  ( $j = 1, 2, \dots$ ) satisfy  $[(n+1)^2\delta + \beta^2]a_n = \sigma_n(\alpha, \beta, \gamma, \delta)$ . If  $n+1 + \beta^2 = 0$ , then for  $j = 1, 2, \dots$  where  $25 + \beta^2 \neq 0$ , the series (13) depends on  $\beta$  and  $\delta$ . If  $a_0 = 0$ , then (13) yields  $w = 0$ . In (13),  $a_0$  is determined by  $k^2\delta + \beta^2 = 0$ . We have three cases: 1)  $w(0) = 0$ ; 2)  $w(0) = w'(0) = 0$ ; 3)  $a_{n-1} = 0$ . If  $\beta \neq 0$  and  $\delta = 0$ , then  $a_j$  are determined with  $a_0 \neq 0$ . For  $j = 1, 2, \dots$  in (13), if  $\delta = 0$ , then for system (1),  $a_{n-1} = 0$ . In case 3), at  $z = 0$ , if  $k^2\delta - \beta^2 = 0$  and  $a_{2s} = 0$  for  $\alpha, \gamma, \delta$  in (1), let  $z = x^s$  ( $s = 0, 1, \dots$ ). Then  $w(z) = hz^3 + 2\alpha/z$  (15). The equation  $ww'' = \tau w'^2 - ww' + 9\tau^2(\alpha w^s + \gamma u - \sqrt{\gamma}x^s w_1 u^2) = 3[(\alpha-1)w_1 + \delta\tau^3 u + \tau^3 u w_1]$  holds. At  $z = 0$ , following N. A. Lukashovich, (15) and (16) imply that if  $a_0 \neq 0$ , then  $a_j$  depends on  $\alpha, \beta, \gamma, \delta$ . If  $a_4 = a_1 = 0$ , then  $a_j = 0$  for  $j > 4$ . If  $a_3, \delta$  are parameters of (1), then at  $z = 0$ , if  $a_0 = 0$ ,  $a_j$  depends on  $\alpha, \beta, \gamma, \delta$ . If  $z = 0$ ,  $x = a_2 = 0$ , then  $a_1 \neq 0$ . In (16), if  $a_2 = a_3 = 0$ , then  $a_2 \neq 0$  and  $a_j, a_3 v = 0$ , meaning  $a_j = 0$ . If  $\beta = \delta = 0$  and  $a_1 \neq 0$ , then for (1), at  $z = 0$  according to [1, 4],  $w(z) = 1/z$  and  $v(z)$  satisfies (2). We have  $u = zu'' - uu' - \gamma zv^2 - \alpha uv$ . At  $z = 0$ , we obtain  $zw'' = zv' - vv' + \delta zu^2 + \beta uv$ . For  $z_0 \neq 0$ ,  $w(z) = \pm \frac{1}{\sqrt{\gamma}(z-z_0)}$ . Following N. A. Lukashovich, with  $h(z-z_0)$  and (7), system (1) yields  $\Gamma = -1/w + zw^2 - \gamma w^2$ . From (7) and (1), if  $w_1$  is known, then  $w_2$  satisfies  $D(w_1, w'_1, w''_1) = (\delta + w_2)z^2 w''_1 + zw'_1 w'_1 - z^2 w_1 w''_1 + 8zw'_1 + \sqrt{\gamma}z^2 w'_1 + \beta - \alpha(\dots)$ . From (23), for  $w, u, w_1, w_2$  as in [5], the dimension is  $d = 4$ . If  $w$  is known, then  $D(w_2, w'_2, w''_2) = 0$  in (23). From (22), we find  $d = 4$ . If  $D(w_2, w'_2, w''_2) = 0$ , then for (1), system (1) with (23) yields  $\delta zw^2$  and  $zw - 2w_1 - \beta \neq 0$ . In the case  $\sqrt{\gamma}w_1 + \delta z + zw_1 zw' + \dots$ , we have  $zw' + 1 - \dots + 2w_2 - \beta$ . If  $w_1(z)$  satisfies (1) and  $D(w_2, w'_2, w''_2) = 0$ , then from (23),  $w_1(z_0) = w_{10}$ . For (1),  $w(z) - \rho(z_0) = w_{10}$ ,  $w(z_0) = w'(z_0) = w'_0$ . From (24) and (25), if (28) and (29)

hold, then  $-\sqrt{\gamma}w_1 + (1 - \dots)z_0(w_2w_0)$ . If  $w_2(z)$  satisfies  $D(w_2, w'_0, w''_0) = 0$  with  $w_2(z_0)$  and  $w'(z_0)$ , then  $w(z)$  satisfies (26) and (27). If (1) holds, then from (32),  $w_{20} = w'_0 + \sqrt{\gamma}w^2 + (\dots + 1)w_0v_{20} = z_0w'_{20} + (\dots + 2)a_{20} - \beta$  for (1). In (7), at  $z = z_0$ ,  $w_1(z) = \frac{-\sqrt{\gamma}}{(z-z_0)^2} + \frac{1}{z-z_0}$ . At  $z_0$ ,  $w(z) = \dots$  and  $w_1(z)$  at  $z = z_0$ . Following N. A. Lukashevich, (21) in (22) shows that  $w_2(z)$  at  $z_0$  has a pole. Thus, if  $\gamma \neq 0$ ,  $\alpha - 2w_1 - \beta \neq 0$ , and  $(\dots + 2)w_2 - \delta \neq 0$ , then for (1) at  $z = 0$ , we have the expansion. We find  $zw'_1 - (\dots - 2)w_1 - \beta = 0$ . From (7), we see that  $zw'_1$  satisfies the condition where  $\sqrt{\gamma} + \delta = 0$ . Given  $\rho, \gamma, \delta$  and  $\rho \pm (w''_0)\sqrt{\delta} = \alpha$ , then  $zw'(\dots + 2)w_2 - \beta = 0$  and  $w_2\sqrt{\delta} \pm (w + 2)\sqrt{\delta} = 0$ . From (36) and (37), we have  $\rho + \dots - \delta = 0$  and  $\gamma = \alpha^2 \neq 0$ . From (38) and (7) and (22), according to [2],  $z = \alpha^*zw^* - (w + b^*z)$ . In the case where  $\gamma = 0$  and  $\alpha \neq 0$ , at  $z = z_0 \neq 0$ , system (1) is consistent with [5], so  $d = 4$ . Substituting (19) into (1), we get:  $zu^2vv'' - zv^2uu' + zv^2u'' - zu^2v'^2 + u^2vv' - v^2uu' - \alpha v^3u - \beta vu^3 - \gamma zv^4 - \delta zu^4 = 0$ . Let  $v(z) = \sqrt{\gamma}u(z) \neq 0$ . In (40) and (39), for  $z$ : a)  $n = m$ ; b)  $\gamma = \alpha = 0$ ; c)  $n < m$  and  $\beta = \delta = 0$ . In cases b) and c), we have  $\alpha = \pm k\sqrt{\gamma}$ . If  $k > 0$  is not an integer and  $\delta = 0$ ,  $\alpha\gamma \neq 0$ , then (1) has the solution  $w(z) = Cz^{1+k}$ . In case a),  $n = m - 0$  for (1), so  $w = 0$ ,  $\delta = 0$ , and  $w = \pm\sqrt{-\beta} \neq 0$  with  $\beta^2 + \alpha^2 = 0$ . For  $z$  and  $w$  in terms of  $\beta, \gamma, \delta$ , if  $\phi < 0$  and  $\beta^2 + \delta\alpha^2 = 0$ , then  $|u\gamma(z_0)| < n = m = 1$ , i.e.,  $v = z + a$  and  $u = bz + c$  at  $z = 0$ . For  $a, b, c$ , if  $b^4 = 0$ ,  $b \neq \beta$ ,  $b^3 + 4\gamma a + 4bb^3c = 0$ ,  $b \neq \alpha$ ,  $c + 3\beta b^2c + \beta ab^3 + 6\alpha^2\gamma + 6b\sqrt{\delta} = 0$ ,  $\alpha^2 - 3c - 3\beta bc^2 - 3\beta ab^2c - 4\gamma\alpha^3 - 4\delta bc^3 = 0$ , and  $\alpha^3b + \alpha^2c + \beta c^3 + 3\beta abc^2 + \gamma\alpha^4 + \delta c^4 = 0$ , then  $(c - ab - \alpha a^2 - \beta c^2)\alpha c = 0$ . Following N. A. Lukashevich, if  $c \neq 0$ , then: 1)  $\alpha = (4b\delta - \beta)b^2$ ; 2)  $\alpha = 0$  from (41); 3)  $\alpha\gamma = -6b^4$  and  $c = 0$ . We have  $S = (2ab + A)\delta$ , where  $S = \beta, \gamma, \epsilon$  are determined by (41). If  $b^2(3c - ab)\epsilon$  satisfies (1) and  $w(z)$  is such that  $a = 0$  and  $c \neq 0$ , then for  $\beta \neq 0$ , we obtain  $9\gamma - a^2 = 0$ ,  $b' + \beta^2 = 0$ , and  $b^2 = -c$ .

In the instance where  $a \neq 0$  and  $c = 0$ , system (1) yields  $\gamma - a^2 = 0$ . Furthermore, if  $9b + a = 20$  and  $\beta^2 = 0$ , we find  $b^2 = -\dots$ ,  $a^2 = -\dots$ , and  $\beta = 0$ . Let us introduce the substitutions  $u(z) = \xi(z) \exp g(z)$  and  $v(z) = \eta(z) \exp g(z)$ , where  $\xi(z)$  and  $\eta(z)$  satisfy the conditions derived from (43) and (20), and  $g(z)$  is a function such that  $\exp 2g(z)$  relates to the system dynamics. The resulting differential equations for  $\eta$  are:

$$z\eta\eta'' = z\eta a'^2 - \eta\eta' - [g'(z) + zg''(z)]\eta^2 + \delta z\xi^2 + \beta g\eta$$

By substituting (45) and (44) into the above, we obtain  $g'(z) + zg''(z) = 2\lambda z + \mu$ , which implies  $g(z) = \lambda z + \mu$ , where  $\lambda$  and  $\mu$  are constants.

The parameters  $\alpha, \beta, \gamma, \delta$  are determined by the consistency of system (1) with equations (45) and (46). For the coefficients  $a_n$  and  $b_n$ , we derive the following recurrence relations:

$$\begin{aligned} z^{2n} + (2Ka_n^2 + \gamma b_1^2) &= 0 \\ b_1^2 + Sa_1^2 &= 0 \\ n_{-1} + 2\gamma &= -(\mu a_n)a_{n-1} - 4\lambda b = (-\beta a\mu b_n)b_n \end{aligned}$$

Assuming  $b_n \neq 0$ , these satisfy (47) and (1). We then find  $\lambda = \pm\sqrt{-\gamma}$  and  $4\lambda^2 + \gamma^8 = 0$ . For  $k = 2, 3, \dots$ , the general form is:

$$z^{2n+1-k} 4ka_n a_{n-k} + 2\gamma b_n b_{n-k} = -P_k(n, a_{n-1}, a_{n-k+1}, b_n, b_{n-k+1})$$

where  $P_k$  is a polynomial function of the indicated variables. The coefficients  $a_j$  and  $b_j$  are determined for all  $j = 0, 1, 2, \dots$  by (51). If  $n \neq 1$  and  $b \neq 0$ , the parameters of (1) are constrained such that  $\gamma = 0$  and  $w(z)$  can be expressed in terms of  $z$ .

### References

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*Note: Figure translations are in progress. See original paper for figures.*

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