ON THE GENERALIZED TRICOMI PROBLEM

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Abstract

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MATHEMATICS

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Let $K(y)$ be a function continuously differentiable on the interval $y_0 \leqslant y \leqslant y_1$, $y_0 < 0$, $y_1 > 0$, having the following properties: $K(y) < 0$ for $y_0 \leqslant y < 0$; $K(y) > 0$ for $0 < y \leqslant y_1$; $K(0) = 0$, $K'(0) > 0$.

Consider in the plane $\{x, y\}$ a domain Q bounded by the closed curve $\Gamma = \Gamma_0 \cup \Gamma_1 \cup \Gamma_2$. The curve Γ_0: $x = x(s)$, $y = y(s)$, $0 \leqslant s \leqslant S$, $x(0) = 1$, $x(S) = 0$, $y(0) = 0$, $y(S) = 0$ (s is the arc length along this curve) is situated in the half-plane $y \geqslant 0$ and has only two common points with the axis Ox, $(0,0)$ and $(1,0)$; $\max_{0\leqslant s\leqslant S} y(s) = y_1$. The functions $x(s)$ and $y(s)$ will be assumed differentiable on $[0, S]$, and the curve Γ_0 to have no singular points. Moreover, suppose that $\dot{x}(0)\dot{y}(0) < 0$, and if $\dot{y}(S) = 0$, then $\dot{x}(S) < 0$. The curve Γ_1 is situated in the half-plane $y \leqslant 0$, passes through the origin, and has the equation $x = \mu(y)$, $y \in [y_0, 0]$, where $\mu(y)$ is continuous for $y_0 \leqslant y \leqslant 0$, differentiable for $y_0 \leqslant y < 0$; the derivative of the function $\mu(y)$ for $y_0 \leqslant y < 0$ satisfies the inequality $\mu'(y) \leqslant -\sqrt{-K(y)}$. The curve Γ_2 is also situated in the half-plane $y \leqslant 0$ and has the equation $x = 1 - \int_y^0 \sqrt{-K(\eta)}\, d\eta$, i.e., passes through the point $(1,0)$. The curves Γ_1 and Γ_2 intersect at the point (ν, y_0), $0 < \nu < 1$.

Consider in the domain Q Chaplygin' s equation

$$T_\lambda(u) \equiv K(y)\partial^2 u/\partial x^2 + \partial^2 u/\partial y^2 + \alpha(x,y)u_x + \beta(x,y)u_y + \\ +\gamma(x,y)u - \lambda u = f(x,y), \qquad (1)$$

where $\alpha(x,y)$, $\beta(x,y)$ are continuously differentiable, and $\gamma(x,y)$ is a continuous function in $\overline{Q}$, λ is a real parameter. The function $f(x,y) \in L_2(Q)$. We note that the curve Γ_2 is a characteristic of equation (1), while the curve Γ_1 has characteristic directions only for those y for which $\mu'(y) = -\sqrt{-K(y)}$.

By the generalized Tricomi problem (1)–(2) we mean the problem of finding in Q a solution of equation (1) satisfying the boundary conditions

$$u\big|_{\Gamma_0 \cup \Gamma_1} = 0. \qquad (2)$$

Denote by $\mathring{W}_2^1(Q)$ the Hilbert space of functions given in Q, obtained by completion in the norm $W_2^1(Q)$ of the set of functions smooth in $\overline{Q}$ and satisfying conditions (2).

Definition. A generalized solution from $\mathring{W}_2^1(Q)$ of problem (1)–(2) is a function $U(x,y) \in \mathring{W}_2^1(Q)$ such that the integral identity

$$\iint_Q [K(y)u_x v_x + u_y v_y - (\alpha u_x + \beta u_y)v + \lambda uv]\,dx\,dy = -\iint_Q fv\,dx\,dy \quad (3)$$

is satisfied for every function $v(x,y) \in \mathring{W}_2^1(Q)$.

Theorem 1. *Under the assumptions made concerning the coefficients of equation (1) and the boundary* Γ *of the domain* Q, *for every function* $f(x,y) \in \widetilde{W}_2^{-1}$ *there exists a unique generalized solution of problem (1)–(2), provided only that* $\lambda \geq \lambda_0$, *where* λ_0 *is some real number.*

Since $L_2(Q) \subset \widetilde{W}_2^{-1}(Q)$, Theorem 1 implies, under its assumptions, the unique solvability of problem (1)–(2) for every function $f(x,y) \in L_2(Q)$.

The principal role in the proof of this theorem is played by the following

Lemma 1. *There exists a constant* $C > 0$ *such that for any function* $U \in W_2^2(Q) \cap \widetilde{W}_2^1(Q)$, *for sufficiently large* λ, *the inequality*

$$\iint_Q (u_{xx}^2 + u_y^2 + u^2)\,dx\,dy + \int_0^1 u^2\big|_{y=0}\,dx + \int_{\Gamma_2} \left(\frac{\partial u}{\partial s}\right)^2 ds + \int_{\Gamma_2} u^2\,ds +$$

$$+ \int_{\Gamma_0} (\dot{x}^2(S) + A(S-s)) \left(\frac{\partial u}{\partial n}\right)^2 ds \leq C \iint_Q (Ku_{xx} + u_{yy} - \lambda u)^2\,dx\,dy. \quad (4)$$

If the curve Γ_1 *has no characteristic points, then to the left-hand side of inequality (4) one may add the term* $\int_{\Gamma_1} \left(\frac{\partial u}{\partial n}\right)^2 ds$; *if, on the curve* Γ_1, *there is a set* E *of characteristic points with* $\operatorname{mes} E < \operatorname{mes} \Gamma_1$, *then to the left-hand side of (4) one may add the term*

$$A_0(\sigma) \int_{\Gamma_1 \setminus E_\sigma} \left(\frac{\partial u}{\partial n}\right)^2 ds,$$

where E_σ, $\sigma > 0$, *is an open set containing the set* E, *the distance from whose boundary to the set* E *is equal to* σ, $\operatorname{mes} E_\sigma < \operatorname{mes} \Gamma_1$, *the constant* $A_0(\sigma) > 0$, $A_0(\sigma) \to 0$ *as* $\sigma \to 0$; $A > 0$.

We outline the proof of this lemma. Let $\delta > 0$ be a sufficiently small number, whose smallness is determined only by the behavior of the function $K(y)$ near

$y = 0$ and by the behavior of the functions $x(s)$ and $y(s)$ near $s = 0$ and $s = S$. Draw in the domain $Q^+ = Q \cap (y > 0)$ a smooth curve Λ, lying in the strip

$$0 < y \leq \max(y(\delta),\, y(S-\delta))$$

and passing through the points $(x(\delta), y(\delta))$ and $(x(S-\delta), y(S-\delta))$. We may assume that the curve Λ adjoins the curve Γ_0 at the points $(x(\delta), y(\delta))$ and $(x(S-\delta), y(S-\delta))$ so smoothly that the boundary of the domain Q_δ, lying between the curves Λ and Γ_0, has a smooth inward normal at these points. Let

$$Q'_\delta = Q^+ \setminus Q_\delta, \qquad Q^- = Q \cap (y < 0),$$

and let $\Gamma_{0,\rho}$, $0 < \rho < S/2$, be the part of the curve Γ_0 for which the parameter $s \in [\rho, S-\rho]$.

Construct the functions $a(x, y)$ and $b(x, y)$ as follows: in the domain

$$Q^- \cup Q'_\delta$$

$$a(x, y) = \varepsilon^{5/4} + \varepsilon^\theta(1-x), \qquad 0 < \theta < 1/4, \qquad b(x, y) = \varepsilon(x/2 - 1)$$

for some sufficiently small $\varepsilon > 0$, while in the domain Q_δ, a and b are solutions of the equations

$$\Delta^2 a = \Delta^2 b = 0,$$

satisfying the boundary conditions: on the contour Λ,

$$a\big|_\Lambda = \varepsilon^{5/4} + \varepsilon^\theta(1-x)\big|_\Lambda, \quad b\big|_\Lambda = \varepsilon(x/2-1)\big|_\Lambda, \quad \partial a/\partial n\big|_\Lambda = -\varepsilon^\theta \cos(\mathbf{n}, \mathbf{x})\big|_\Lambda,$$

$$\partial b/\partial n\big|_\Lambda = \varepsilon \frac{\cos(\mathbf{n}, \mathbf{x})}{2}\bigg|_\Lambda, \quad \text{on the contour } \Gamma_{0,\delta} \quad a\big|_{\Gamma_{0,\delta}} = -\dot{y}(s), \quad b\big|_{\Gamma_{0,\delta}} = \dot{x}(s).$$

On the portions of the curves $\Gamma_{0,\delta}/\Gamma_{0,2\delta}$, the functions $a(x, y)$ and $b(x, y)$ are obtained by linear interpolation between the corresponding values of $a(x, y)$ and $b(x, y)$ at the points

$$(x(\delta), y(\delta)), \quad (x(2\delta), y(2\delta)), \quad (x(S-\delta), y(S-\delta)), \quad (x(S-2\delta), y(S-2\delta)),$$

$$\frac{\partial a}{\partial n}\bigg|_{\Gamma_{0,\delta}} = g(s), \qquad \frac{\partial b}{\partial n}\bigg|_{\Gamma_{0,\delta}} = g_1(1),$$

where $g_1(s)$ and $g(s)$ are arbitrary smooth functions on the interval $\delta < s < S-\delta$, taking at the endpoints of this interval the values

$$g(\delta) = -\varepsilon^\theta \cos(\mathbf{n}, \mathbf{x})\big|_{\Gamma_0,\,(s=\delta)},$$

$$g(S-\delta) = -\varepsilon^\theta \cos(\mathbf{n}, \mathbf{x})\big|_{\Gamma_0\,(s=S-\delta)}, \qquad g_1(\delta) = g(\delta)\frac{\varepsilon^{1-\theta}}{2}, \qquad g_1(S-\delta) = g(S-\delta)\frac{\varepsilon^{1-\theta}}{2}.$$

By $c(x,y)$ we denote a function equal to a sufficiently large constant $C_0 > 0$ in the domain Q^+ and equal to

$$\frac{C_0 K'(0)}{4\sqrt[4]{-K(y)}} \int_y^0 \frac{e^{n\eta}\, d\eta}{\sqrt[4]{-K^3(\eta)}}$$

for sufficiently large $n > 0$ in the domain Q^- (the latter integral is convergent, since $K(y)$ is a smooth function and $K'(0) \neq 0$).

Applying the Friedrichs a, b, c method, i.e., multiplying equation (1) for $\alpha = \beta = \gamma = 0$ by $au_x + bu_y + cu$, with the a, b, and c just constructed, and integrating the equality thereby obtained over the domain Q, we obtain the required estimate (4). We note that first a sufficiently small $\delta > 0$ is fixed, then $\varepsilon > 0$ is chosen sufficiently small, then $C_0 > 0$ is taken sufficiently large, and only after that is $n > 0$ chosen sufficiently large.

Lemma 1 immediately implies the following.

Lemma 2. If $u(x,y) \in \widetilde{W}_2^2(Q) \cap \dot{W}_2^1(Q)$, then for sufficiently large positive λ

$$\iint_Q (u_x^2 + u_y^2 + u^2)\, dx\, dy \leq C \iint_Q (Ku_{xx} + u_{yy} - \lambda u)\, dx\, dy$$

with a constant $C > 0$ independent of the chosen function u.

The proof of Theorem 1 is then carried out as follows. Denote by R_λ the extension of the operator T_λ (1), under the conditions (2), defined by the integral identity (3). The domain of definition of the operator R_λ is $\widetilde{W}_2^1(Q)$, and its range is the Hilbert space $\dot{W}_2^{-1}(Q)$. It is easy to see that the operator R_λ is bounded from $\widetilde{W}_2^1(Q)$ into $\dot{W}_2^{-1}(Q)$; therefore, by Lemma 2, for $u \in \widetilde{W}_2^1(Q)$ and sufficiently large λ the inequality

$$\|u\|_{\widetilde{W}_2^1(Q)} \leq C \|R_\lambda u\|_{\dot{W}_2^{-1}(Q)} \tag{5}$$

holds, and if $R_\lambda u \in L_2(Q)$, then also the inequality

$$\|u\|_{\widetilde{W}_2^1(Q)} \leq C \|R_\lambda u\|_{L_2(Q)}.$$

From the last inequality follows the uniqueness theorem, for large λ, for the generalized solution of the problem (1)–(2). Since the set $\{R_\lambda u\}$, $u \in \widetilde{W}_2^1(Q)$, is obviously closed, in order to prove solvability, for sufficiently large λ, of the problem (1)–(2), it is enough to establish that the orthogonal complement in $\dot{W}_2^{-1}(Q)$ to $\{R_\lambda u\}$, $u \in \widetilde{W}_2^1(Q)$, is empty. The latter is also derived from (3) and (5).

Further, by known methods ([3]) one can obtain the proof of Theorem 2.

Theorem 2. The problem of finding a generalized solution to the problem (1) –(2) for arbitrary λ is Fredholm.

We note that a number of important results on the generalized Tricomi problem for certain special cases of equation (1) were obtained in the works ([1,2,4–6]).

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Note: Figure translations are in progress. See original paper for figures.