Bounded and oscillatory solutions of the system $\ddot{x} + g(x,\, \dot{x}) = \bar{0}$

Authors: L. V. Khokhlova

Date: 1967-01-01T00:00:00+00:00

Abstract

A series of theorems is proved regarding the topological structure of the phase portrait of a system of ordinary differential equations

$$\left.\begin{aligned} \dot{x} &= y \\ \dot{y} &= -g(x,y) \end{aligned}\right\} \tag{1}$$

which the author calls relaxational if: (I) the function $g(x,y)$ is defined and continuous on the $2n$-dimensional real Euclidean space R^{2n}, where $g(x,\bar{0}) \neq \bar{0}$ for $x \neq \bar{0}$, $g(\bar{0},\bar{0}) = \bar{0}$, and for a piecewise smooth arc $\overset{\smile}{0x}$ in R^n (connecting points $\bar{0}$ and x), the line integral

$$G(x) \equiv \int_{\overset{\smile}{0x}} g(x,\bar{0})\, dx \begin{cases} > 0 & x \neq 0, \\ \equiv 0 & x = \bar{0} \end{cases}$$

increases indefinitely with the chord $|x|$; (II) for a function y defined and continuous on $\overset{\smile}{0x}$, satisfying $ydx \equiv |y||dx|$,

$$\lim_{x \to +\infty} \int_{\overset{\smile}{0x}} \{g(x,y) - g(x,\bar{0})\}\, dx > -\infty$$

wherever the differential dx exists. Bibliography: 11 items.

Full Text

Preamble

This study, submitted in 1967, investigates the qualitative behavior of solutions to the n-dimensional differential equation:

$$\ddot{x} + g(x, \dot{x}) = 0$$

which can be written as the system:

$$\begin{cases} \dot{x} = y \\ \dot{y} = -g(x, y) \end{cases} \tag{1}$$

where $x, y \in \mathbb{R}^n$. We assume that $g(x, y)$ is a continuous function on $\mathbb{R}^{2n}$ such that $g(0, 0) = 0$. Following the classical works of Liénard [**?**, **?**], we define the potential function $G(x)$ for the case $n = 1$ as $G(x) = \int_0^x g(\xi, 0)d\xi$. For $n = 3$, the condition rot $g(x, 0) = 0$ is required [**?**], and for $n > 3$, the existence of a scalar potential $G(x)$ such that $g(x, 0) = \operatorname{grad} G(x)$ is assumed [**?**].

We introduce the following fundamental conditions: (I) $G(x) \to \infty$ as $|x| \to \infty$, where $G(x) > 0$ for $x \neq 0$ and $G(0) = 0$. (II) The dissipative term $R(x, y) = [g(x, y) - g(x, 0)] \cdot y$ satisfies $R(x, y) \geq 0$ for all $x, y \in \mathbb{R}^n$.

These conditions generalize the classical Van der Pol oscillator properties to higher dimensions. For instance, in the case of the scalar Liénard equation $\ddot{x} + \mu(x^2 - 1)\dot{x} + x = 0$, the function $G(x) = x^2/2$ satisfies condition (I), while the term $R(x, y) = \mu(x^2 - 1)y^2$ satisfies condition (II) only outside the unit interval $|x| < 1$.

1. Global Existence and Boundedness

Let $M(t) = \{x(t), y(t)\}$ be a solution to system (1) defined on the maximal forward interval $[0, \tau)$. We define the energy function:

$$E(x, y) = G(x) + \frac{1}{2}|y|^2 \tag{5}$$

Along the trajectories of (1), the derivative of the energy is given by:

$$\dot{E} = -R(x, y) \tag{15}$$

If condition (II) holds ($R \geq 0$), the energy $E(t)$ is non-increasing. This implies that if the initial energy is finite, the solution remains in a bounded region of the phase space $\mathbb{R}^{2n}$ for all $t \in [0, \tau)$, which further implies that the solution can be extended indefinitely ($\tau = +\infty$).

Theorem 1. If conditions (I) and (II) are satisfied, then every solution of (1) is defined for all $t \geq 0$ and is bounded in $\mathbb{R}^{2n}$.

In cases where condition (II) is violated in some bounded region $\Omega \subset \mathbb{R}^{2n}$ (i.e., $R(x, y) < 0$ for some x, y), we can still establish global existence if the growth of $g(x, y)$ is restricted. Specifically, if there exists a constant $C > 0$ such that:

$$|g(x, y)| \leq C(|y| + 1) \tag{6}$$

then the solutions do not diverge to infinity in finite time.

Theorem 2. Suppose condition (I) holds and there exists a constant $C > 0$ such that for $|y| > N$, the dissipation $R(x, y)$ satisfies $R(x, y) \geq 0$. Then any solution $M(t)$ of (1) that is bounded in x must also be bounded in y, and thus exists for all $t \geq 0$.

2. Stability and Limit Cycles

We now consider the behavior of solutions when the dissipation $R(x, y)$ is negative near the origin and positive far from the origin, a characteristic feature of self-sustained oscillations. Let $\mathcal{D}$ be a bounded domain in $\mathbb{R}^{2n}$ containing the origin. We assume: (III) $g(x, y)$ is locally Lipschitz continuous. (IV) There exists a constant $g > 0$ such that for $G(x) > g$, the dissipation $R(x, y)$ is strictly positive. (V) For sufficiently small $|y|$, the origin is unstable.

Under these conditions, we can apply the Poincaré-Bendixson theory (or its n-dimensional generalizations) to prove the existence of limit cycles. If the energy $E(x, y)$ increases near the origin and decreases far from the origin, there must exist at least one invariant set (a limit cycle or a more complex attractor) within the region where the energy balance shifts.

Theorem 3. If conditions (I), (III), (IV), and (V) are satisfied, then system (1) possesses at least one stable limit cycle in $\mathbb{R}^{2n}$.

3. Applications to Multidimensional Systems

The results obtained can be extended to systems of the form:

$$\ddot{x}_k + f_k(x_k)\dot{x}_k + x_k = 0, \quad k = 1, \dots, n \tag{45}$$

where each component represents a Liénard-type oscillator. If the functions $f_k(x_k)$ satisfy the classical Liénard conditions (e.g., $f_k(x_k) < 0$ for $|x_k| < a$ and $f_k(x_k) > 0$ for $|x_k| > a$), then the total system exhibits complex oscillatory behavior. The energy function for the aggregate system is the sum of the individual energies:

$$E(x, y) = \sum_{k=1}^{n} \left(\frac{1}{2} y_k^2 + \int_0^{x_k} \xi d\xi \right)$$

The dissipation is $R(x, y) = \sum f_k(x_k) y_k^2$. Our theorems ensure that such multidimensional systems remain bounded and, under appropriate conditions on f_k, converge to a limit set representing synchronized or multi-frequency oscillations.

References

1. Liénard A. *Etude des oscillations entretenues.* Revue Générale de l’ Electricité, 1928, v. 23, p. 901–912.
2. Liénard A. *Proc. of the Third Int. Congr. of Appl. Mech.* (Stockholm, 1930), v. 3, p. 173–177.
3. Sansone G., Conti R. *Nonlinear Differential Equations.* Pergamon Press, 1964.
4. Lefschetz S. *Differential Equations: Geometric Theory.* Interscience Publishers, 1957.
5. Barbashin E. A. *Introduction to the Theory of Stability.* Nauka, 1967.

6. Van der Pol B. *Relaxation-Oscillations.* Philosophical Mag., 1926, v. 2 (7th ser.), p. 978.
7. Nemytskii V. V., Stepanov V. V. *Qualitative Theory of Differential Equations.* Gostekhizdat, 1947.
8. Krasovskii N. N. *Certain Problems in the Theory of Stability of Motion.* Fizmatgiz, 1959.
9. Pontryagin L. S. *Ordinary Differential Equations.* Nauka, 1965.
10. Reissig R., Sansone G., Conti R. *Qualitative Theorie Nichtlinearer Differentialgleichungen.* Edizioni Cremonese, 1963.
11. Khapaev M. M. *On the stability of equilibrium states.* Doklady Akademii Nauk SSSR, 1966, v. 170 (112), p. 140–149.

Note: Figure translations are in progress. See original paper for figures.