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Abstract

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MATHEMATICS

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AN EQUATION OF PARABOLIC TYPE WITH A PERIODIC COEFFICIENT

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The subject of the present note is the Cauchy problem for the parabolic equation

$$\begin{gathered} \partial u/\partial t = \partial^2 u/\partial x^2 + q(x,\tau)u, \qquad \tau = \nu t, \\ Au(x,t) \equiv a_1 u'(0,t) + a_2 u(0,t) = 0, \\ Bu(x,t) \equiv b_1 u'(1,t) + b_2 u(1,t) = 0, \\ \operatorname*{l.i.m.}_{t\to 0} u(x,t) = \psi(x), \end{gathered} \tag{1}$$

in the half-strip Π $(0 \le x \le 1,\ t > 0)$. The numbers a_1, a_2, b_1, and b_2 are real, with $a_1^2 + a_2^2 > 0$ and $b_1^2 + b_2^2 > 0$; $\psi(x)$ is an arbitrary quadratically integrable function; ν is a positive parameter. We shall be interested in those properties of problem (1) that it possesses for sufficiently large values of the parameter ν.

Regarding the function $q(x,\tau)$ the following is assumed. It is real and periodic with period 2π, expands in an absolutely convergent Fourier series

$$q(x,\tau) = \sum_{k=-\infty}^{\infty} q_k(x) e^{ik\tau}$$

with continuous coefficients, and has a quadratically integrable derivative with respect to t. In addition, it is assumed that there exists a sufficiently small positive number σ such that the series

$$\|q\|_\sigma = \max_{0\le x\le 1} |q_0(x)| + \sum_{k=-\infty}^{\infty} |k|^\sigma \max_{0\le x\le 1} |q_k(x)|$$

converges.

We shall agree to call a **Floquet solution** of problem (1) any function $u(x,t)$ satisfying the first three relations in (1) and representable in the form

$$u(x,t) = e^{\gamma t} v(x,t), \tag{2}$$

where $v(x,t)$ is a periodic function with period $T = 2\pi/\nu$.

Denote by Λ_μ $(0 \leq \mu \leq 1)$ the totality of all periodic functions $y(x,t) = y(x,t+T)$ whose Fourier coefficients are continuous and satisfy the condition: the series

$$\|y\|_\mu = \max_{0\leq x\leq 1} |y_0(x)| + \sum_{k=-\infty}^{\infty}{}' |k|^\mu \max_{0\leq x\leq 1} |y_k(x)| \tag{3}$$

converges. If $\|y\|_\mu$ is taken as the norm of the function $y(x,t)$, then Λ_μ becomes a Banach space.

Theorem 1. *If ν is greater than a certain number $\nu_m(\|q\|_0)$, depending only on the magnitude of $\|q\|_0$,*

$$\nu > \nu_m(\|q\|_0),$$

then problem (1) has at least m Floquet solutions $u_k(x,t)$ $(k = 1, 2, \dots, m)$, belonging to Λ_1.

Denote by L the operator

$$L = d^2/dx^2 + q_0(x),$$

considered on functions satisfying the boundary conditions $A\varphi = 0$, $B\varphi = 0$. Let $\varphi_k(x)$ and λ_k $(k = 1, 2, \dots)$ be a complete set of eigenfunctions and eigenvalues of the operator L, with $\lambda_{k+1} < \lambda_k$. The eigenfunctions $\varphi_k(x)$ have norm equal to one in $\mathcal{L}_2(0,1)$, and are uniformly bounded:

$$\int_0^1 \varphi_k^2(x)\,dx = 1, \qquad \max_{0\leq x\leq 1} |\varphi_k(x)| < \theta \quad (k = 1, 2, \dots).$$

Theorem 2. If $\nu > \nu_m(\|q\|_0)$ and $\lambda_{m+1} < 0$, then the solution of problem (1) can be represented in the form

$$u(x,t) = \sum_{k=1}^{m} c_k u_k(x,t) + w(x,t); \tag{4}$$

here c_k $(k = 1, 2, \ldots, m)$ are certain coefficients, determined below, and $w(x,t)$ is a function that can be estimated as follows:

$$\left| w(x,t) - \sum_{k=m+1}^{\infty} h_k e^{\lambda_k t} \varphi_k(x) \right| < \tag{5}$$

$$< 2\theta \|h\| \left[1 + 2(\alpha - \lambda_{m+1}) \sum_{k=m+2}^{\infty} (\lambda_{m+1} - \lambda_k)^{-1} \right]^{1/2} e^{\alpha t} F_m \left(Q_1 \sqrt{\frac{2t}{\alpha - \lambda_{m+1}}} \right);$$

h_k are the Fourier coefficients of the function

$$h(x) = \psi(x) - \sum_{k=1}^{m} c_k u_k(x, 0);$$

α is an arbitrary number from the interval $\lambda_{m+1} < \alpha \leq 0$;

$$q_1(x,\tau) \equiv q(x,\tau) - q_0(x), \qquad Q_1 = \max_{x,\tau} |q_1(x,\tau)|; \tag{6}$$

$\|h\|$ denotes the norm of the function $h(x)$ in $\mathcal{L}_2(0,1)$, and, finally,

$$F_m(z) \equiv \sum_{n=m+1}^{\infty} \frac{z^n}{\sqrt{n!}}.$$

The function $w(x,t)$ has a continuous derivative with respect to t in the half-strip Π.

Let us note that the right-hand side of inequality (5) for $\alpha = 0$ decreases as the index m grows proportionally to $|\lambda_m|^{-(m+1)/2}$ uniformly in t $(t \geq 0)$. On the other hand, with a suitable choice of the parameter α, the right-hand side does not exceed the quantity

$$M e^{(\lambda_{m+1} + 2Q_1^2 + \varepsilon)t}$$

for all $t \geq 0$, if $\varepsilon > 0$ is any positive number and M is a sufficiently large number.

Denote by $z_k(x,t) \in \Lambda_1$ $(k = 1, 2, \ldots, m)$ the Floquet solutions of problem (1) in which the function $q(x,t)$ is replaced by the function $q(x,-t)$. The functions $z_k(x,t)$ may be renumbered and normalized so that

$$\int_0^1 u_i(x,t) z_k(x,-t)\, dx = \delta_{ik} \qquad (i, k = 1, 2, \ldots, m).$$

Theorem 3. The coefficients c_k $(k = 1, 2, \dots, m)$ appearing in relations (4) and (5) can be computed by the formula

$$c_k = \int_0^1 \psi(x) z_k(x, 0)\, dx.$$

The following two theorems make it possible to find Floquet solutions $u_k(x, t)$ and the function $w(x, t)$.

Theorem 4. We represent the Floquet solution $u_k(x, t)$ $(k = 1, 2, \dots, m)$ in the form (2). The function $v(x, t)$ is the limit in the norm Λ_1 of the sequence $v_n(x, t)$, defined as follows:

1. $v_0(x, t) \equiv \varphi_k(x)$.
2. If the function $v_n(x, t)$ $(n = 0, 1, \dots)$ has already been defined, then the function $v_{n+1}(x, t)$ is found as the solution in Λ_1 of the problem

$$\partial v_{n+1}/\partial t = (L - \lambda_k) v_{n+1} + [q_1(x, t) - \gamma_n + \lambda_k] v_n, \tag{7}$$

$$A v_{n+1} = 0, \qquad B v_{n+1} = 0, \qquad A^+ v_{n+1,0} \equiv a_1 v_{n+1,0}(0) - a_2 v'_{n+1,0}(0) = A^+ \varphi_k(x),$$

where

$$\gamma_n = \lambda_k + \int_0^1 \varphi_k(x) \big(q_1(x, t) v_n(x, t)\big)_0\, dx \Big/ \int_0^1 \varphi_k(x) v_{n,0}(x)\, dx. \tag{8}$$

All the numbers γ_n determined in this way are finite, and the sequence of these numbers tends to γ. The rate of approximation of the function $v(x, t)$ by the functions $v_n(x, t)$ is characterized by the following estimates:

$$\left| v_{n+1,0}(x) - v_{n,0}(x) \right| < N_0 \left(\frac{a_0}{\nu} \right)^{1 + [n/2]},$$

$$\left\| v_{n+1,1}(x, t) - v_{n,1}(x, t) \right\|_0 < N_1 \left(\frac{a_1}{\nu} \right)^{1 + \left[\frac{n+1}{2} \right]} \qquad (n = 0, 1, \dots). \tag{9}$$

Here N_0, N_1, a_0, and a_1 are some constants independent of ν.

The indices zero and one in equalities (7), (8), and (9) have the same meaning as in definition (6).

Theorem 5. The function $w(x, t)$ is equal to

$$w(x,t) = w_0(x,t) + \sum_{n=1}^{\infty} w_n(x,t); \tag{10}$$

the series on the right-hand side converges uniformly in Π, and the functions $w_n(x,t)$ $(n = 0, 1, ...)$ are defined as follows:

$$w_n(x,t) = \sum_{k=1}^{m} c_k^{(n)}(t)\varphi_k(x) + \xi_n(x,t) \qquad (n = 0, 1, ...), \tag{11}$$

$$\xi_0(x,t) = \sum_{k=m+1}^{\infty} h_k e^{\lambda_k t}\varphi_k(x), \qquad \xi_n(x,t) = \sum_{k=m+1}^{\infty} \varphi_k(x) \int_0^t e^{\lambda_k(t-s)} f_{n-1,k}(s)\,ds,$$

$$(n = 1, 2, ...), \tag{12}$$

$f_{n,k}(s)$ are the Fourier coefficients of the function $q_1(x,\nu s)w_n(x,s)$ with respect to the functions $\varphi_k(x)$ $(k = 1, 2, ...)$. The coefficients $c_k^{(n)}(t)$ are determined from the conditions

$$\int_0^1 w_n(x,t)z_k(x,-t)\,dx = 0 \qquad (t \geq 0,\ k = 1, 2, \dots, m;\ n = 0, 1, ...). \tag{13}$$

The terms of the series (10) can be estimated as follows:

$$|w_n(x,t)| \leq 2\theta\|h\| \left[1 + 2(a - \lambda_{m+1}) \sum_{k=m+2}^{\infty} (\lambda_{m+1} - \lambda_k)^{-1}\right]^{1/2} \times$$

$$\times \frac{e^{at}}{\sqrt{n!}} \left[\frac{2Q_1^2 t}{\alpha - \lambda_{m+1}}\right]^{n/2}, \tag{14}$$

where a is any number from the interval $(\lambda_{m+1} < a \leq 0)$.

The function $w(x,t)$ is continuously differentiable with respect to t in Π. Its derivative can be obtained by termwise differentiation of the series (10). The estimate

$$|\dot{w}_n(x,t)| \leq 2\sigma_n(t_0)(t/t_0)^{3/4n}(1 - t_0/t)^{-1/3} \qquad (n = 1, 2, ...),$$

holds, where $t_0 < t$ is an arbitrary positive number, and $\sigma_n(t_0)$ is a collection of functions such that

$$\lim_{n\to\infty} \sqrt[n]{\sigma_n(t_0)} = 0.$$

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Note: Figure translations are in progress. See original paper for figures.