Solution by the method of lines of certain boundary value problems for equations of elliptic type

Authors: V. T. Ivanov

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Abstract

This paper considers the solution of certain boundary value problems for a two-dimensional elliptic equation with coefficients depending on two variables using the method of lines. The general solution to the system of differential equations of the method of lines is sought using the method proposed by V. I. Lebedev. To find the eigenvalues and eigenvectors of the finite-difference boundary value problem, the theory of Jacobi determinants and matrices is employed. An explicit formula for the eigenvectors allows for the formulation of the general solution to the system of differential equations, provided that the general solution of n uncoupled ordinary differential equations and a particular solution of the non-homogeneous system of the method of lines are known. In conclusion, the well-posedness of the method of lines for solving the third boundary value problem for elliptic equations is established. Bibliography: 7 items.

Full Text

Preamble

In 1967, V. T. Khorovtsev [1] investigated boundary value problems for second-order partial differential equations. Consider the following elliptic equation:

$$a_1(x)\frac{\partial^2 u}{\partial x^2} + a_2(x)\frac{\partial u}{\partial x} - a_3(x)u + a_0(x)\frac{\partial^2 u}{\partial y^2} = f(x, y)$$

subject to the boundary conditions:

$$\alpha_1 u(x, 0) - \beta_1 \frac{\partial u(x, 0)}{\partial y} = \phi_1(x), \quad \alpha_2 u(x, b) + \beta_2 \frac{\partial u(x, b)}{\partial y} = \phi_2(x)$$

where $\beta_i > 0$ $(i = 1, 2)$. For discrete values $y = y_k$ $(k = 1, 2, \dots, n)$, the problem can be approximated by a system of ordinary differential equations:

$$a_1(x)u_k'' + a_2(x)u_k' - a_3(x)u_k + a_0(x)\frac{1}{h^2}(B_k u_{k+1} - A_k u_k + C_k u_{k-1}) = f_k(x)$$

where h is the step size in y. The boundary conditions at $x = 0$ and $x = a$ are given by:

$$\alpha_3 u(0, y) - \beta_3 u'_x(0, y) = \phi_3(y), \quad \alpha_4 u(a, y) + \beta_4 u'_x(a, y) = \phi_4(y)$$

with $\beta_i > 0$ and $\alpha_i + \beta_i > 0$ for $i = 3, 4$.

Numerical Analysis and Stability

To solve the system of equations (7)-(8), we employ a separation of variables approach. Let $u_k(x) = y(k)v(x)$. Substituting this into the homogeneous part of the system leads to the eigenvalue problem:

$$B_k y(k+1) + (\lambda - A_k)y(k) + C_k y(k-1) = 0$$

with boundary conditions $Y(0) = \alpha Y(1)$ and $Y(n+1) = \beta_{n+1} Y(n)$. This results in a tridiagonal matrix whose characteristic equation $D_n(\lambda) = 0$ determines the eigenvalues λ_j. Under the conditions $B_k > 0$ and $C_k > 0$, the eigenvalues λ_j are real and distinct [4].

The general solution for each mode $v_j(x)$ satisfies:

$$a_1(x)v''_j + a_2(x)v'_j - (a_3(x) + \lambda_j a_0(x))v_j = 0$$

The complete solution $u_k(x)$ is then constructed as a linear combination of these eigenfunctions, satisfying the boundary conditions (4)-(5).

Convergence and Error Estimation

We define the operator $L_h u_k$ and establish a maximum principle for the discrete system. If $L_h u_k \geq 0$ in the domain $(0, a)$ and u_k is not constant, then the maximum value of u_k must occur on the boundary. This property ensures the stability of the numerical scheme.

Let $\epsilon_k(x)$ be the error of the approximation. The error satisfies the system $L_h \epsilon_k = R_k(x)$, where $R_k(x)$ is the truncation error. Using the properties of the coefficients $a_i(x)$ and the bounds on the derivatives of the exact solution $M_m = \max |\partial^m u/\partial x^m|$, we can estimate the error magnitude. Specifically, if $m = \min[a_3(x) + \lambda]$, the error is bounded by:

$$|\epsilon_k(x)| \leq \frac{1}{12m} \max |a_0(x)(b_{1k}M_4 + 2|b_{2k}|M_3)| + ...$$

This demonstrates that the numerical solution converges to the exact solution as the mesh size h approaches zero, provided the coefficients satisfy the required smoothness and ellipticity conditions [5, 6].

Conclusion

The method of lines described above provides a robust framework for solving elliptic boundary value problems. By reducing the partial differential equation to a system of coupled ordinary differential equations, we leverage well-established techniques for tridiagonal systems and eigenvalue problems. The stability and convergence of the method are guaranteed under the specified constraints on the boundary coefficients and the operator L_h.

References

1. Khorovtsev, V. T. (1967). On certain boundary value problems for elliptic equations.
2. Faddeev, D. K., & Faddeeva, V. N. (1963). *Computational Methods of Linear Algebra*.
3. Givens, W. (1954). *Numerical computation of the characteristic values of a real symmetric matrix*. U.S. Atomic Energy Commission, ORNL-1574.
4. Krein, M. G. (1952). On the theory of symmetric operators. *Progress in Mathematical Sciences*, 30(3), 695–702.
5. Samarskii, A. A. (1965). *Theory of Difference Schemes*.
6. Richtmyer, R. D. (1956). *Difference Methods for Initial-Value Problems*.

Note: Figure translations are in progress. See original paper for figures.