UNSTEADY OUTFLOWS OF GAS IN A CONSTANT GRAVITATIONAL FIELD

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Abstract

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UNSTEADY OUTFLOWS OF GAS IN A CONSTANT GRAVITATIONAL FIELD

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Consider the problem of the one-dimensional outflow of an ideal gas at rest, located in a constant gravitational field and obeying the equation $P = \text{const} \cdot \rho^k$. Previously this problem was solved for one simplest particular case, $k = 3$ ([1]).

The basic equations of the problem will be

$$u_t + uu_x + P_x/\rho = -a, \qquad \rho_t + u\rho_x + \rho u_x = 0, \tag{1}$$

where $a = \text{const}$ is the acceleration due to gravity.

In the case of an ideal gas we shall have

$$u_t + uu_x + \frac{2}{k-1} cc_x = -a, \qquad c_t + uc_x + \frac{k-1}{2} cu_x = 0. \tag{2}$$

We note that the characteristic conditions of system (2) have the form: along the lines $dx/dt = u \pm c$ the relation

$$w \pm \frac{2}{k-1} c = \text{const}, \tag{3}$$

is satisfied, where $w = u + at$.

A gas at rest, compressed by the gravitational field, is described by the equations

$$c^2 = c_{\text{H}}^2 - (k-1)ax, \qquad u = 0, \tag{4}$$

where $c = c_{\text{H}}$ at $x = 0$.

Let, at the time $t = 0$, the outflow of gas begin in the section $x = 0$. Since there is an initial distribution of pressure or speed of sound along the length x, the outflow of gas cannot be described by special solutions of the system of equations. For this purpose it is necessary to use the general solutions of this system.

As was shown by one of the authors ([1]), these solutions have the form

$$\psi = \frac{\partial^{n-1}}{\partial i^{n-1}} \frac{F_1\left[\sqrt{2(2n+1)i+w}\right] + F_2\left[\sqrt{2(2n+1)i-w}\right]}{\sqrt{i}},$$

$$t = \partial\psi/\partial i, \qquad x = x_0 + wt - \frac{at^2}{2} - \partial\psi/\partial w, \tag{5}$$

where

$$x_0 = \text{const}, \qquad k = (2n+3)/(2n+1), \qquad n = 0, 1, 2, \dots \tag{6}$$

It is obvious that, during outflow, the wave-front of rarefaction moving to the left is described by the equation

$$\partial x/\partial t = -c = -\sqrt{c_{\text{H}}^2 - (k-1)ax},$$

whence, integrating under the condition $x = 0,\ t = 0$, we find

$$x = -\left(c_{\text{H}}t + \frac{k-1}{4}at^2\right). \tag{7}$$

On this front, since $u = 0$, we shall have

$$at - \frac{2}{k-1}c = -\frac{2}{k-1}c_{\text{H}} \quad \text{or} \quad w = at = \frac{2}{k-1}(c - c_{\text{H}}), \tag{8}$$

which, incidentally, is verified identically by substituting (7) into (4).

Since (8) can be written in the form

$$\sqrt{2(2n+1)i - w} = \sqrt{2(2n+1)i_n} = \text{const},$$

then

$$F_2[\sqrt{2(2n+1)i-w}] = F_2(\text{const}) = \text{const},$$

and consequently this function cannot be determined from the condition on the left-hand characteristic (7). With the aid of this characteristic one can find the form of the function $F_1[\sqrt{2(2n+1)i+w}]$.

Since on the characteristic

$$t = \frac{\partial \psi}{\partial i} = \frac{2}{(k-1)a}(c - c_n) = \frac{1}{a}\left[\sqrt{2(2n+1)i} - \sqrt{2(2n+1)i_n}\right] \tag{9}$$

and the condition ([1]) holds,

$$\left(\omega \frac{\partial}{\partial \omega}\right)^n \frac{F_1(\omega - w)}{\omega} = \frac{1}{2^n}\frac{\partial^n}{\partial \omega^n}\frac{F_1(2\omega - \beta)}{\omega^{n+1}}, \tag{10}$$

where

$$\omega = \frac{2}{k-1}c = \sqrt{2(2n+1)i}, \qquad \beta = \omega - w = \omega_n = \frac{2}{k-1}c_n = \text{const}, \tag{11}$$

the solution for ψ will take the form

$$\psi = \frac{\partial^{n-1}}{\partial i^{n-1}}\left\{A\left[\sqrt{2(2n+1)i} + w + \sqrt{2(2n+1)i_n}\right]^{2(n+1)}\right.$$

$$\left. - B\omega_n\left[\sqrt{2(2n+1)i} + w + \sqrt{2(2n+1)i_n}\right]^{2n+1} + F_2\right\}/\sqrt{i}. \tag{12}$$

Here, on the characteristic (11), the condition $F_2 = 0$ must be satisfied, and

$$t = \frac{\omega - \omega_n}{a} = \frac{[2(2n+1)]^{n+1/2}}{2^{2n}}\frac{\partial^n}{\partial \omega^n}\frac{A(2\omega)^{2(n+1)} - B\omega_n(2\omega)^{2n+1}}{\omega^{n+1}} =$$

$$= [2(2n+1)]^{n+1/2}\left[4A\frac{d^n\omega^{n+1}}{d\omega^n} - 2B\omega_n\frac{d^n\omega^n}{d\omega^n}\right] =$$

$$= [2(2n+1)]^{n+1/2}\left[4(n+1)!\,A\omega - 2n!\,B\omega_n\right],$$

whence

$$A = \frac{1}{4a(n+1)!\,[2(2n+1)]^{n+1/2}}, \qquad B = \frac{1}{2an!\,[2(2n+1)]^{n+1/2}}.$$

Thus,

$$\psi = \frac{1}{4a(n+1)!\,[2(2n+1)]^{n+1/2}}\frac{\partial^{n-1}}{\partial i^{n-1}}\left\{\left[\sqrt{2(2n+1)i} + w + \right.\right.$$

$$+\sqrt{2(2n+1)i_n}\Big]^{2(n+1)} - 2(n+1)\omega_n \Big[\sqrt{2(2n+1)i}+$$

$$+w+\sqrt{2(2n+1)i_n}\Big]^{2n+1} + F_2\Big\} / \sqrt{i}, \tag{13}$$

or

$$\psi = \frac{1}{4a(n+1)!\,[2(2n+1)]^{n+1/2}} \frac{\partial^{n-1}}{\partial i^{n-1}} \Big\{ \Big[\sqrt{2(2n+1)i} + w +$$

$$+\sqrt{2(2n+1)i_n}\Big]^{2(n+1)} - 2(2n+1)^{1/2}\sqrt{i_n}\cdot 2(n+1)\Big[\sqrt{2(2n+1)i}+$$

$$+w+\sqrt{2(2n+1)i_n}\Big]^{2n+1} + F_2\Big\} / \sqrt{i}. \tag{14}$$

The function ψ can also be written in the form

$$\psi = \frac{1}{4a(n+1)!\,[2(2n+1)]^{n+1}} \frac{\partial^n}{\partial i^n} \left\{ \frac{\Big[\sqrt{2(2n+1)i} + w + \sqrt{2(2n+1)i_n}\Big]^{2n+3}}{2n+3} \right.$$

$$\left. - \omega_n \Big[\sqrt{2(2n+1)i} + w + \sqrt{2(2n+1)i_n}\Big]^{2(n+1)} + F_2 \right\}. \tag{15}$$

For $n = 0$, $x_0 = 0$. In the general case (for $n = 1, 2, \dots$) the value x_0 may prove to be nonzero. To compute x_0 we proceed as follows–

as follows. Since for $u = 0$

$$x = (c^2 - c^2)/(k-1)a = (\omega^2 - \omega^2)/2(2n+1)a,$$

$$x = x_0 + at^2/2 - \partial\psi/\partial w,$$

then hence, comparing the expressions for x, we shall have

$$x_0 = (\omega^2 - \omega^2)/2(2n+1)a - (\omega - \omega\ )^2/2a + \partial\psi/\partial w$$

and after calculation we obtain the expression for $\partial\psi/\partial w$ on characteristic (11).

Since

$$\frac{\partial\psi}{\partial w} = \frac{1}{2an!\,[2(2n+1)]^{n+1/2}}\frac{\partial^{n-1}}{\partial i^{\,n-1}}\left\{[\omega - w + \omega\,]^{2n+1} - \right.$$

$$\left. -(2n+1)\omega\,[\omega + w + \omega\,]^{2n}\right\}/\sqrt{i},$$

then, using condition (10), we may write

$$\frac{\partial\psi}{\partial w} = \frac{2^n}{an!\,2(2n+1)}\frac{\partial^{n-1}}{(\omega\partial\omega)^{n-1}}\frac{2\omega^{2n+1} - (2n+1)\omega\,\omega^{2n}}{\omega} =$$

$$= \frac{2}{an!\,2(2n+1)}\frac{\partial^{n-1}}{\partial\omega^{\,n-1}}\left[2\omega^{n+1} - (2n+1)\omega\,\omega^n\right] =$$

$$= \frac{2\omega}{2(2n+1)a}\left[(n+1)\omega - (2n+1)\omega\,\right],$$

which immediately determines

$$x_0 = -n\omega^2/(2n+1)a. \tag{16}$$

Let us determine the functions F_2, which can be done by assuming that at $t = 0,\ x = 0$ the condition is satisfied

$$w = u = \frac{2}{k-1}(c\ - c) = \omega\ - \omega. \tag{17}$$

Since along the left characteristic (11) $F_2 = 0$, it is obvious that the expression for F_2 must be sought in the form

$$F_2 = F_2\left[\sqrt{2(2n+1)i} - w - \sqrt{2(2n+1)i}\,\right] = F_2[\omega - (w + \omega\,)]\,.$$

Then, when condition (11) is satisfied, $F_2 = F_2(0) = 0$. For $n = 0, 1, 2, \ldots$ one may prescribe F_2 in the form $F_2 = \sum A_r[\omega - (w + \omega\,)]^r$. In this case the value

$$F_1 = F_1[\omega + w + \omega\,] = F_1(2\omega\,) = 2^{2(n+1)}n\omega^{2(n+1)}. \tag{18}$$

Writing the expressions $t = \partial\psi/\partial v = 0$ and $x = x_0 - \partial\psi/\partial w = 0$, or $\partial\psi/\partial w = x_0$, when (17) is satisfied, it is easy for each n to determine A_r and the degree r in the expression for F_2.

For $n = 0,\ F_2 \equiv 0$; for $n = 1, 2, 3$, the calculations are comparatively simple, becoming more complicated as n increases, and already for $n = 4$ they are rather cumbersome. We give the values of ψ for $n = 0, 1, 2, 3, 4$:

$$\psi_{n=0} = \frac{1}{8a}\left[(\omega + w + \omega\ )^3 - 3\omega\ (\omega + w + \omega\ )^2\right]/3;$$

$$\psi_{n=1} = \frac{1}{48\sqrt{6}\,a}\left\{(\omega + w + \omega\ )^4 - 4\omega\ (\omega + w + \omega\ )^3 + \right.$$

$$\left. +4\omega^2[\omega - (w + \omega\ )]^2\right\}/\sqrt{i};$$

$$\psi_{n=2} = \frac{1}{24 \cdot 10^2\sqrt{10}\,a}\frac{\partial}{\partial i}\left\{(\omega + w + \omega\ )^6 - 6\omega\ (\omega + w + \omega\ )^5 + \right.$$

$$\left. +12\omega^2[\omega - (w + \omega\ )]^4 - 8\omega^3[\omega - (w - \omega\ )]^3\right\}/\sqrt{i};$$

$$\psi_{n=3} = \frac{1}{96 \cdot 14^2\sqrt{14a}}\frac{\partial^2}{\partial i^2}\left\{(\omega + w + \omega_{\mathrm{H}})^8 - 8\omega_{\mathrm{H}}(\omega + w + \omega_{\mathrm{H}})^7 + \right.$$
$$+24\omega_{\mathrm{H}}^2[\omega - (w + \omega_{\mathrm{H}})]^6 + 32\omega_{\mathrm{H}}^3[\omega - (w + \omega_{\mathrm{H}})]^5 +$$
$$\left. +16\omega_{\mathrm{H}}^4[\omega - (w + \omega_{\mathrm{H}})]^4\right\}/\sqrt{i};$$

$$\psi_{n=4} = \frac{1}{480 \cdot 18^4\sqrt{18a}}\frac{\partial^3}{\partial i^3}\left\{[\omega + w + \omega_{\mathrm{H}}]^{10} - 10\omega_{\mathrm{H}}(\omega + w + \omega_{\mathrm{H}})^9 + \right.$$
$$+40\omega_{\mathrm{H}}^2[\omega - (w + \omega_{\mathrm{H}})]^8 + 80\omega_{\mathrm{H}}^3[\omega - (w + \omega_{\mathrm{H}})]^7 +$$
$$\left. +80\omega_{\mathrm{H}}^4[\omega - (w + \omega_{\mathrm{H}})]^6 + 32\omega_{\mathrm{H}}^5[\omega - (w + \omega_{\mathrm{H}})]^5\right\}/\sqrt{i}.$$

We see that the number of terms in F_2 is equal to n. The maximum degree is $2n$, the minimum $n+1$.

Thus, in general form one may write

$$\psi = \frac{1}{4a(n+1)!\,[2(2n+1)]^{n+1/2}}\frac{\partial^{n-1}}{\partial i^{n-1}}\left\{(\omega + w + \omega_{\mathrm{H}})^{2(n+1)} - \right.$$
$$\left. -2(n+1)\omega_{\mathrm{H}}(\omega + w + \omega_{\mathrm{H}})^{2n+1} + \sum_{r=n+1}^{2n} A_r(\omega - w - \omega_{\mathrm{H}})^r\right\}\Big/\sqrt{i},$$

where the coefficients A_r, as we have just indicated, are computed algebraically for each n.

The front of the gas rarefaction will obey the law $c = 0$, $w = \dfrac{2}{k-1}c_{\mathrm{H}} = \omega_{\mathrm{H}}$, or $dx/dt = u = \omega_{\mathrm{H}} - at$, whence $x = \omega_{\mathrm{H}}t - at^2/2$; at $t = \omega_{\mathrm{H}}/a$, $x = \omega_{\mathrm{H}}^2/2a$, the gas reaches the maximum point of ascent and begins to fall "downward." In this case a new wave arises, which can be found simply, since in it

$$u = 0, \qquad c^2 = \frac{2}{k-1} c_{\mathrm{H}}^2 - (k-1)ax. \tag{19}$$

Near the point $u = 0$, $c = 0$, for example for $k = 5/3$, there will be the asymptotic expression

$$w = (\omega_{\mathrm{H}} - \omega)/2 + \frac{1}{2}\sqrt{\omega_{\mathrm{H}}^2 + 9\omega^2 - 2\omega\omega_{\mathrm{H}}} = \omega_{\mathrm{H}} - \omega + 2\omega^2/\omega_{\mathrm{H}}.$$

The problem considered is also of practical and methodological interest, since for the very first rarefaction wave one has to determine both arbitrary functions F_1 and F_2, which does not occur in ordinary problems of unsteady gas dynamics.

In conclusion it should be noted that if we solved the problem of the motion of a gas with constant acceleration, then the solutions of the equations for this problem would be equivalent to (5), but the initial conditions and the wave system would be different, since the initial gas density is the same everywhere.

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REFERENCES

1. K. P. Stanyukovich, *Unsteady Motions of a Continuous Medium*, 1955, § 76.

Note: Figure translations are in progress. See original paper for figures.