ON THE INTERPOLATION OF FUNCTIONS REGULAR ON CLOSED SETS

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Abstract

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MATHEMATICS

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ON THE INTERPOLATION OF FUNCTIONS REGULAR ON CLOSED SETS

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In this note estimates are obtained for certain quantities arising in the study of the measure of uniform approximation of functions regular on a closed set by means of interpolation polynomials. In the exposition we adhere to the terminology and notation used in (^{1}), Ch. I, §§ 1-3. Some definitions and notation are given below.

A **set of type** $\mathfrak{M}$ will mean a closed set of points of the complex plane, containing more than one point, whose complement (in the extended plane) is a simply connected domain containing the point ∞.

Let: $\overline{B}$ be a set of type $\mathfrak{M}$; B^1 its complement; L the boundary of B^1; $w = \varphi(z)$ a function conformally and univalently mapping B^1 onto the domain $|w| > 1$, with $\varphi(\infty) = \infty$, $\varphi'(\infty) = 1/c > 0$; $z = \psi(w)$ the inverse function for $w = \varphi(z)$; L_ρ, $\rho > 1$, the image of the circle $|w| = \rho$ under the mapping $z = \psi(w)$; B_ρ the bounded domain with boundary L_ρ; B^1_ρ the unbounded domain with boundary L_ρ.

Let $z_1^{(n)}, z_2^{(n)}, \dots, z_{n+1}^{(n)}$, $n = 0, 1, \dots$, be a sequence of interpolation nodes situated on L_r, $1 < r < \infty$.

Put $w_k^{(n)} = \varphi(z_k^{(n)})$. The nodes $z_k^{(n)}$, $k = 1, 2, \dots, n+1$, will be called **equally spaced on** L_r if the points $w_k^{(n)}$ divide the circle $|w| = r$ into $n+1$ equal parts. In what follows we assume that the nodes $z_k^{(n)}$ are equally spaced on L_r, $1 < r < \infty$. Below we estimate the function

$$\omega_n(z) = \prod_{k=1}^{n+1} \left(z - z_k^{(n)}\right)$$

and certain functions connected with it.

First we formulate a result of N. A. Lebedev (see (^{2}), Ch. IV, § 2, Theorem 3), concerning the class Σ of functions $F(\zeta) = \zeta + a_0 + a_1/\zeta + \cdots$, regular and univalent in the domain $1 < |\zeta| < \infty$.

Lemma 1. Let $F(\zeta) \in \Sigma$; let γ_ν and γ'_ν, $\nu = 1, 2, \dots, m$, be arbitrary numbers, and let ζ_ν and ζ'_ν, $\nu = 1, 2, \dots, m$, be arbitrary points in the domain $1 < |\zeta| < \infty$. Then

$$\left| \sum_{\nu,\nu'=1}^{m} \gamma_\nu \overline{\gamma_{\nu'}} \ln \frac{F(\zeta_\nu) - F(\zeta'_{\nu'})}{\zeta_\nu - \zeta'_{\nu'}} \right| \leq$$

$$\leq \left[\sum_{\nu,\nu'=1}^{m} \gamma_\nu \overline{\gamma_{\nu'}} \ln \left(1 - \frac{1}{\zeta_\nu \overline{\zeta_{\nu'}}} \right) \sum_{\nu,\nu'=1}^{m} \gamma'_\nu \overline{\gamma'_{\nu'}} \ln \left(1 - \frac{1}{\zeta'_\nu \overline{\zeta'_{\nu'}}} \right) \right]^{1/2}. \qquad (1)$$

Here are understood those branches of

$$\ln \frac{F(\zeta) - F(\xi)}{\zeta - \xi} \quad \text{and} \quad \ln \left(1 - \frac{1}{\zeta \overline{\xi}} \right),$$

which vanish at zero for $\zeta = \infty$.

Relying on Lemma 1, we shall prove the following lemma.

Lemma 2. For every $z = \psi(w)$, $|w| > 1$, and every $\zeta = \psi(\xi)$, $|\xi| > 1$, the inequalities

$$\left| \ln \frac{\omega_n(z)}{c^{n+1}(w^{n+1} - \eta_n)} \right| \leq \left[(n+1) \ln \frac{|w|^2}{|w|^2 - 1} \ln \frac{r^{2(n+1)}}{r^{2(n+1)} - 1} \right]^{1/2}, \qquad \eta_n = (w_k^{(n)})^{n+1}, \qquad (2)$$

$$\left| \ln \frac{\omega_n(z)(\xi^{n+1} - \eta_n)}{\omega_n(\zeta)(w^{n+1} - \eta_n)} \right| \leq \left[(n+1) \ln \frac{|w\overline{\xi} - 1|^2}{(|w|^2 - 1)(|\xi|^2 - 1)} \ln \frac{r^{2(n+1)}}{r^{2(n+1)} - 1} \right]^{1/2}. \quad (3)$$

In particular, for $\zeta = z_k^{(n)}$ we obtain

$$\left| \ln \frac{\omega_n(z)(n+1)\varphi'(z_k^{(n)})\eta_n}{\omega'_n(z_k^{(n)})(w^{n+1} - \eta_n)w_k^{(n)}} \right| \leq \left[(n+1) \ln \frac{|ww_k^{(n)} - 1|^2}{(|w|^2 - 1)(|w_k^{(n)}|^2 + 1)} \ln \frac{r^{2(n+1)}}{r^{2(n+1)} - 1} \right]^{1/2}. \qquad (4)$$

Proof. Let us note that the function $F(\zeta) = \frac{1}{c}\psi(\zeta) \in \Sigma$. To prove inequality (2), we apply inequality (1) to this function $F(\zeta)$, putting $m = n + 1$; $\gamma_1 = 1$; $\gamma_\nu = 0$, $\nu = 2, \dots, n + 1$; $\gamma'_\nu = 1$, $\nu = 1, 2, \dots, n + 1$; $\zeta_1 = w$; $\zeta'_\nu = w_\nu^{(n)}$, $\nu = 1, 2, \dots, n + 1$. We then obtain

$$\left|\ln \frac{\omega_n(z)}{c^{n+1}(w^{n+1} - \eta_n)}\right| \le \left[\ln\left(1 - \frac{1}{|w|^2}\right) \sum_{\nu'=1}^{n+1} \sum_{\nu=1}^{n+1} \ln\left(1 - \frac{1}{w_\nu^{(n)} \overline{\overline{w_{\nu'}^{(n)}}}}\right)\right]^{1/2}. \quad (5)$$

But

$$\sum_{\nu=1}^{n+1} \ln\left(1 - \frac{1}{w_\nu^{(n)} w_{\nu'}^{(n)}}\right) = \ln \prod_{\nu=1}^{n+1}\left(1 - \frac{1}{w_\nu^{(n)} w_{\nu'}^{(n)}}\right) = \ln\left(1 - \frac{1}{r^{2(n+1)}}\right).$$

Hence (2) follows from (5). Inequality (3) is proved similarly, but one must put $m = n + 1$; $\gamma_1 = 1$; $\gamma_2 = -1$; $\gamma_\nu = 0$, $\nu = 3, \dots, n + 1$; $\gamma'_\nu = 1$, $\nu = 1, 2, \dots, n + 1$; $\zeta_1 = w$; $\zeta_2 = \xi$; $\zeta'_\nu = w_\nu^{(n)}$, $\nu = 1, 2, \dots, n + 1$. Inequality (4) is obtained if in (3) we let ζ tend to $z_k^{(n)}$. The lemma is proved.

Further, using the following fact: if $|\ln B| \le A$, then $|B - 1| \le Ae^A$, we obtain:

Corollary 1. If $z \in B_\rho^1$, $r < \rho < \infty$, then

$$\left|\frac{\omega_n(z)}{[c\varphi(z)]^{n+1}} - 1\right| \le \left(\frac{r}{\rho}\right)^{n+1} + \left[1 + \left(\frac{r}{\rho}\right)^{n+1}\right] A_n(\rho) e^{A_n(\rho)},$$

$$\left|\frac{\sqrt[n+1]{\omega_n(z)}}{c\varphi(z)} - 1\right| \le \frac{r^{n+1}}{(n+1)(\rho^{n+1} - r^{n+1})} + \frac{2}{n+1} A_n(\rho) e^{\frac{1}{n+1}A_n(\rho)},$$

where

$$A_n(\rho) = \left[(n+1) \ln \frac{\rho^2}{\rho^2 - 1} \ln \frac{r^{2(n+1)}}{r^{2(n+1)} - 1}\right]^{1/2}.$$

Corollary 2. If $z \in L_\rho$, $1 < \rho < r$, then

$$\left|\frac{\omega_n(z)}{(z - z_k^{(n)})\omega'_n(z_k^{(n)})} - \frac{w_k^{(n)} \psi'(w_k^{(n)})}{(z_k^{(n)} - z)(n+1)}\right| \le$$

$$\le \frac{1}{n+1} \left|\frac{w_k^{(n)} \psi'(w_k^{(n)})}{z_k^{(n)} - z}\right| \left\{\left(\frac{\rho}{r}\right)^{n+1} + \left[1 + \left(\frac{\rho}{r}\right)^{n+1}\right] B_n(\rho) e^{B_n(\rho)}\right\},$$

where

$$B_n(\rho) = \left[(n+1)\ln\frac{(\rho r+1)^2}{(\rho^2-1)(r^2-1)}\ln\frac{r^{2(n+1)}}{r^{2(n+1)}-1}\right]^{1/2}. \tag{6}$$

We note that for $z = \psi(w) \in L_\rho,\ 1 < \rho < r$ (see (1), Ch. I, § 2, Lemma 2) we have

$$\left|\frac{w_k^{(n)}\psi'(w_k^{(n)})}{z_k^{(n)}-z}\right| \leqslant \frac{r+\rho}{r-\rho}.$$

Let now the set $\overline{B}$ be a closed bounded simply connected domain whose boundary L is an analytic curve. Then there exists a number $r_0,\ 0 < r_0 < 1$, such that the function $z = \psi(w)$, regular and univalent in the domain $r_0 < |w| < \infty$ (see (2), Ch. II, § 3, Theorem 5) and the lines L_ρ, may be considered for $r_0 < \rho < \infty$. Let the nodes $z_k^{(n)} = \psi(w_k^{(n)})$ be equidistributed on L. Introduce the function $z = \psi_*(w) = \psi(r_0 w)$. For this function, as before, one can introduce the lines $L_\rho^*,\ 1 < \rho < \infty$, and moreover $L_\rho^* = L_{\rho/r_0}$, and the nodes will be equidistributed on L_{1/r_0}. Applying now, for example, Corollary 2, we obtain:

Corollary 2^1**.** Let B be a bounded simply connected domain whose boundary L is a closed analytic curve, and let the function $z = \psi(w)$ be regular and univalent in the domain $r_0 < |w| < \infty,\ 0 < r_0 < 1$. If the nodes $z_k^{(n)}$ are equidistributed on L, then for $z \in L_\rho,\ r_0 < \rho < 1$, the inequality holds

$$\left|\frac{\omega_n(z)}{(z-z_k^{(n)})\omega_n'(z_k^{(n)})} - \frac{w_k^{(n)}\psi'(w_k^{(n)})}{(z_k^{(n)}-z)(n+1)}\right| \leqslant$$

$$\leqslant \frac{1}{n+1}\left|\frac{w_k^{(n)}\psi'(w_k^{(n)})}{z_k^{(n)}-z}\right|\left[\rho^{n+1} + (1+\rho^{n+1})B_n^*(\rho)e^{B_n^*(\rho)}\right],$$

where

$$B_n^*(\rho) = \left[(n+1)\ln\frac{(\rho+r_0^2)^2}{(\rho^2-r_0^2)(1-r_0^2)}\ln\frac{1}{1-r_0^{2(n+1)}}\right]^{1/2}.$$

The possible applications are clear from what was set forth in (1), Ch. 1, §§ 1, 2, 3, 6. Let us give an example.

From inequality (3) there follows directly the inequality

$$\left|\frac{\omega_n(z)}{\omega_n(\zeta)}\right| \leqslant \frac{2r^{n+1}}{\rho_1^{n+1}-r^{n+1}}e^{B_n(\rho)}, \qquad z\in L_r, \quad \zeta\in L_\rho, \quad 1<r<\rho,$$

where $B_n(\rho)$ is determined by formula (6). Hence, and from (1), Ch. I, § 2, Lemma 3, there immediately follows

Theorem. Let $\overline{B}$ be a set of type $\mathfrak{M}$. If the function $f(z)$ is regular in the domain B_{ρ_0}, $1<\rho_0<\infty$, and

$$\frac{1}{2\pi}\int_0^{2\pi}\left|f(\psi(\rho e^{i\theta}))\right|d\theta \leqslant M, \qquad 1<\rho<\rho_0,$$

then for the interpolation polynomials $P_n(z)$, $n=1,2,\ldots$, constructed with respect to nodes equidistributed on L_r, $1<r<\rho_0$, the estimates hold

$$|f(z)-P_n(z)| \leqslant M\frac{\rho_0+r}{\rho_0-r}\,\frac{2r^{n+1}}{\rho_0^{n+1}-r^{n+1}}e^{B_n(\rho)}, \tag{7}$$

where $B_n(\rho_0)$ is determined by formula (6). For $z\in\overline{B}$ the fraction $(\rho_0+r)/(\rho_0-r)$ may be replaced by $(\rho_0+1)/(\rho_0-1)$.

This theorem is analogous to Theorem 3 from (1), Ch. I, § 3, 1°.

If the conditions of this theorem are fulfilled and it is necessary to estimate $|f(z)-p_n(z)|$ only on $\overline{B}$, then for each n we may assign the nodes $z_k^{(n)}$ on some line L_{r_n}, $1<r_n<\rho_0$, chosen so that the right-hand side of inequality (7) is minimal, or as close as possible to minimal.

Set

$$r_n = \sqrt[2(n+1)]{(n+1)\ln(n+1)} = e^{\frac{1}{2}\ln[(n+1)\ln(n+1)]/(n+1)}.$$

Since $1<r_n<\rho_0$, n must satisfy the inequality

$$\ln[(n+1)\ln(n+1)]/2(n+1) < \ln\rho_0. \tag{8}$$

In this case

$$B_n(\rho_0) = \left[(n+1)\ln\frac{(\rho_0 r_n+1)^2}{r_n(\rho_0^2-1)(r_n-r_n^{-1})}\ln\left(1+\frac{1}{r_n^{2(n+1)}-1}\right)\right]^{1/2} \leq$$

$$\leq \left[(n+1)\ln\frac{(\rho_0^2+1)^2(n+1)}{\rho_0(\rho_0^2-1)\ln[(n+1)\ln(n+1)]}\,\frac{1}{(n+1)\ln(n+1)+1}\right]^{1/2}. \tag{9}$$

It is easy to see that, when the inequality

$$e^{-1/(n+1)} \ln[(n+1)\ln(n+1)] \geq \frac{(\rho_0^2+1)^2}{\rho_0(\rho_0^2-1)} \tag{10}$$

is satisfied, the right-hand side of inequality (9) does not exceed unity, and inequality (7) takes the form

$$|f(z) - p_n(z)| \leq 2e\frac{\rho_0+1}{\rho_0-1} M \frac{[(n+1)\ln(n+1)]^{1/2}}{1 - \dfrac{[(n+1)\ln(n+1)]^{1/2}}{\rho_0^{n+1}}} \frac{1}{\rho_0^{n+1}},$$

where n satisfies conditions (8) and (10).

This inequality is a strengthening of the inequality given in (1), Ch. I, § 3, 1°, Remark 4.

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CITED LITERATURE

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