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Abstract

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A DOUBLY PERIODIC PROBLEM IN THE THEORY OF A CIRCULAR CLOSED CYLINDRICAL SHELL

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The paper formulates the problem of the state of stress in a circular cylindrical shell weakened by a doubly periodic system of identical circular holes. A solution is constructed for the system of equations of the theory of cylindrical shells in the class of doubly periodic functions.

1. Let the exterior of a system of congruent circles on the surface of the shell form a doubly periodic lattice with fundamental periods $\omega_1' = 2a$ and $\omega_2' = 2ale^{i\alpha}$ $(0 < \alpha \leq \pi/2)$. Denote the contour of the hole with center at the point $P' = m\omega_1' + n\omega_2'$ $(m, n = 0, \pm 1, ...)$ by $L_{m,n}$; place the origin of coordinates at the center of the hole $L_{0,0}$; direct the x-axis along a generator, and the y-axis along the directrix of the cylinder. For simplicity we shall assume that the lattice is symmetric with respect to the coordinate axes and is loaded along the edges of the holes by a symmetric system of forces and moments, and is also subjected to the action of a uniform internal pressure and a uniform tension at infinity. Introducing dimensionless coordinates $x = \bar{x}/a$, $y = \bar{y}/a$, we write the system of equations of the theory of a circular cylindrical shell in the form

$$\nabla^2\nabla^2 F_1 = \varepsilon \frac{\partial^2 F_2}{\partial x^2}, \qquad \nabla^2\nabla^2 F_2 = -\varepsilon \frac{\partial^2 F_1}{\partial x^2} + \frac{q}{D}, \tag{1}$$

$$\varepsilon = \frac{a^2}{Rh}\sqrt{12(1-\mu^2)}, \qquad D = \frac{Eh^3}{12(1-\mu^2)}, \qquad U = F_1, \quad w = \frac{\sqrt{12(1-\mu^2)}}{Eh^2} F_2;$$

R and h are the radius and thickness of the shell; E and μ are the Young' s modulus and Poisson's ratio of the shell material; q is the intensity of the normal load; U and w are the stress and deflection functions in the shell.

The problem consists in constructing a doubly periodic function F_2 and a quasiperiodic function F_1, satisfying the system of equations (1), the boundary conditions of the first fundamental problem on the boundary $L = \bigcup L_{m,n}$, the static conditions within the period parallelogram, and the condition of closure of the shell along a generator. In addition, the single-valuedness conditions for the displacements ([1]) must be satisfied.

2. Introduce the representations

$$F_s = F_s^0 + F_s^*, \qquad F_s^0 = \operatorname{Re}\{\bar{z}\varphi_s(z) + \chi_s(z)\}, \tag{2}$$

$$F_s^* = \sum_{m,n} d_{m,n}^{(s)}\xi, \qquad s = 1, 2,$$

where ([2])

$$\varphi_s(z) = A_s z + \sum_{k=0}^{\infty} \alpha_{2k+2}^{(s)} \frac{\lambda^{2k+2}\mathfrak{P}^{(2k-1)}(z)}{(2k+1)!}, \qquad \alpha_2^{(2)} = \beta_2^{(2)} = B_2 = 0,$$

$$\chi_s(z) = (s-1)B_0 + \frac{1}{2}B_s z^2 + \sum_{k=0}^{\infty} \frac{\lambda^{2k+2}}{(2k+1)!}\left\{\beta_{2k+2}^{(s)}\mathfrak{P}^{(2k-2)}(z) - \alpha_{2k+2}^{(s)}Q^{(2k-1)}(z)\right\},$$

$$\wp(z) = \frac{1}{z^2} + \sum_{m,n}{}' \left\{\frac{1}{(z-P)^2} - \frac{1}{P^2}\right\}, \qquad Q(z) = \sum_{m,n}{}' \left\{\frac{\overline{P}}{(z-P)^2} - 2z\frac{\overline{P}}{P^3} - \frac{\overline{P}}{P^2}\right\},$$

$$z = x + iy, \qquad P = m\omega_1 + n\omega_2, \qquad \overline{P} = m\omega_1 + n\bar{\omega}_2, \qquad \omega_1 = 2,$$

$$\omega_2 = 2le^{i\alpha}, \qquad \alpha_m = \pi m/\alpha, \qquad \beta_n = \pi n/\beta, \qquad \xi = \cos\alpha_m x \cos\beta_n y,$$

$\wp(z)$ is the Weierstrass elliptic function ([3]); $Q(z)$ is a special meromorphic function ([2,4]); 2λ is the dimensionless diameter of the hole; α and β are the half-periods along the axes x and y, respectively; the constants $\alpha_{2k+2}^{(s)}$ and $\beta_{2k+2}^{(s)}$ are to be determined from the boundary conditions; the constants $d_{m,n}^{(s)}$ are to be determined from system (1).

The condition of double periodicity of F_2 is satisfied automatically. The static conditions on arcs connecting congruent points z with $z + \omega_1$ and z with $z + \omega_2$ give

$$A_1 = \frac{a^2}{4}(N_1 + N_2) + (K_0\alpha_2^{(1)} + K_1\beta_2^{(1)})\lambda^2,$$

$$B_1 = \frac{a^2}{2}(N_2 - N_1) + (K_2\alpha_2^{(1)} + K_3\beta_2^{(1)})\lambda^2, \tag{3}$$

where

$$K_1 = \pi i/(\overline{\omega}_1\omega_2 - \overline{\omega}_2\omega_1), \qquad K_2 = (\gamma_1 - \delta_1)/\omega_1 + 4K_1, \qquad \delta_1 = 2\zeta(\omega_1/2),$$

$$K_0 = K_3 = \delta_1/\omega_1 - 2K_1, \qquad \gamma_1 = 2Q(\omega_1/2) - \overline{\omega}_1\wp(\omega_1/2),$$

$\zeta(z)$ is the Weierstrass zeta-function, and N_1 and N_2 are components of the basic stress state.

The tangential displacements u and v in the middle surface of the shell are found by integrating Hooke' s law with allowance for (2). We have

$$v = \frac{1}{2Gha}\,\mathrm{Im}\{x\varphi_1(z) - z\overline{\Phi_1'(z)} - \overline{\psi_1(z)}\} - \frac{a\varepsilon^*}{R}\,\mathrm{Im}\left\{\int \chi_2\,dz + \overline{z}\int \varphi_2\,dz + \iint \varphi_2\,dz^2\right\} - \frac{a\varepsilon^*}{R}\sum_{m,n} d_{m,n}^{(2)}\frac{\eta^*}{\beta_n} + B(x), \tag{4}$$

where

$$\eta^* = \cos\alpha_m x \sin\beta_n y, \qquad \varepsilon^* = \sqrt{12(1-\mu^2)/Eh^2}, \qquad G = E/2(1+\mu).$$

The condition for closure of the shell is the periodicity of the displacement v along the directrix of the cylinder:

$$v(z + \omega_2) - v(z) \equiv 0. \tag{5}$$

From (5) and (2), taking into account the properties of the introduced functions (2) and formulas (3), we find

$$\varepsilon B_0 = a^2(N_2 - \mu N_1) + \frac{\pi\lambda^2}{H\omega_1}(2\alpha_2^{(1)} + \beta_2^{(1)}) +$$

$$+ \frac{\varepsilon\lambda^4}{6H\omega_1}\{H(\gamma_1 - \delta_1)\alpha_4^{(2)} + (\delta_1 H - \pi)\beta_4^{(2)}\}, \tag{6}$$

where $2H = 2l\sin\alpha$.

Thus, the condition of shell closure along the generator does not affect the stress state in the shell. In this respect the formulation under discussion differs from other analogous formulations, in which the shell-closure condition cannot be fulfilled, indicating their known incorrectness. Finally, the condition of uniqueness of the displacement v, as is easily seen from formula (4), is satisfied automatically.

3. To integrate system (1), we represent the functions F_s^0, by means of σ-multipliers ([5]), in the form of double trigonometric series and substitute them, together with the functions F_s^*, into equations (1). After carrying out the necessary

formal operations, we obtain

$$F_1^* = -\varepsilon\sum_{k=0}^{\infty}\left\{\alpha_{2k+2}^{(2)}\Pi_k + \beta_{2k+2}^{(2)}\Pi_k^* + \varepsilon\alpha_{2k+2}^{(1)}(\Omega_k + \nu_1\delta_k^0\Omega) + \varepsilon\beta_{2k+2}^{(1)}(\Omega_k^* + \nu_2\delta_k^0\Omega)\right\},$$

$$F_2^* = \varepsilon\sum_{k=0}^{\infty}\left\{\alpha_{2k+2}^{(1)}(\Pi_k + \nu_1\delta_k^0\Pi) + \beta_{2k+2}^{(1)}(\Pi_k + \nu_2\delta_k^0\Pi) - \varepsilon\alpha_{2k+2}^{(2)}\Omega_k - \varepsilon\beta_{2k+2}^{(2)}\Omega_k^*\right\}, \tag{7}$$

where

$$\Pi_k = \frac{\lambda^{2k+2}}{(2k+1)!}\frac{d^{2k}\Pi_0}{dx^{2k}}, \qquad \Pi_k^* = \frac{\lambda^{2k+2}}{(2k+1)!}\frac{d^{2k}\Pi_0^*}{dx^{2k}}, \qquad \Omega_k = \frac{\lambda^{2k+2}}{(2k+1)!}\frac{d^{2k}\Omega_0}{dx^{2k}},$$

$$\nu_1 = K_0 + \frac{K_2}{2},$$

$$\Omega_k^* = \frac{\lambda^{2k+2}}{(2k+1)!}\frac{d^{2k}\Omega_0^*}{dx^{2k}}, \qquad \Pi_0 = \sum_{m,n}\chi_{m,n}A_{m,n}\xi,$$

$$\Pi_0^* = \sum_{m,n}\chi_{m,n}a_{m,n}\xi, \qquad \nu_2 = K_1 + \frac{K_3}{2},$$

$$\Omega_0 = \sum_{m,n}\delta_{m,n}A_{m,n}\xi, \qquad \Omega_0^* = \sum_{m,n}\delta_{m,n}a_{m,n}\xi, \qquad \chi_{m,n} = \frac{\alpha_m^2(\alpha_m^2+\beta_n^2)^2}{\Delta}, \qquad \delta_{m,n} = \frac{\alpha_m^4}{\Delta},$$

$$\Delta = \varepsilon^2\alpha_m^4 + (\alpha_m^2 + \beta_n^2)^2, \qquad A_{m,n} = b_{m,n} - 2ka_{m,n}, \qquad \delta_k^0 = \begin{cases} 1, & k = 0, \\ 0, & k \neq 0. \end{cases}$$

The quantities $b_{m,n}$ and $a_{m,n}$ are the Fourier coefficients of the functions $\mathrm{Re}\{-\overline{z}\xi(z) - \int Q(z)\,dz\}$ and $\mathrm{Re}\{\int \xi(z)\,dz\}$, respectively.

Representations (2) and (7) determine the general solution of system (1). The derivation, from the boundary conditions on the contour of the hole $L_{0,0}$, of the infinite system of linear algebraic equations for the quantities $\alpha_{2k+2}^{(s)}$ and $\beta_{2k+2}^{(s)}$ presents no difficulty.

In conclusion, let us note that the method developed here makes it possible to consider a considerably broader class of problems. In particular: a finite cylindrical shell with one or several holes, a doubly periodic or periodic problem for a spherical shell, a perforated shell under the action of concentrated forces, etc. Special cases of the problem considered here are shells with one hole, with a row of holes along the generator, or along the directrix.

Formulas (4) and (6) make it possible to give a rigorous formulation of the problem of reducing a cylindrical doubly periodic lattice to a continuous shell equivalent to it in stiffness.

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Note: Figure translations are in progress. See original paper for figures.