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Abstract

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MATHEMATICS

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SOME RESULTS IN SET-THEORETIC TOPOLOGY

(Presented by Academician P. S. Aleksandrov on 1 IV 1966)

In this note we shall state our results without proofs. The methods of proof belong to the so-called combinatorial set theory and, first of all, to the theory of "partition relations" of Erdős and Rado (see, for example, ([6])).

§ 1. In this section we deal with the cardinality of discrete subspaces of topological spaces. As is known, recently de Groot ([1]), B. A. Efimov ([2]), and J. Isbell ([3]) considered this question and proved, independently of one another, similar results. We have succeeded in improving these results and proving others.

Theorem 1. *Every Hausdorff space R of cardinality greater than* $\exp\exp\mathfrak{m}$ *contains a discrete subspace of cardinality greater than* $\mathfrak{m}$*. (Here* $\mathfrak{m}$ *is an arbitrary infinite cardinal number.)*

This theorem follows at once from the set-theoretic lemma 1.

Lemma 1. *Let H be an arbitrary set, and suppose that to each element $x \in H$ there is assigned some system $S(x)$ of subsets of the set H such that: a) from $U, V \in S(x)$ it follows that $U \cap V \in S(x)$; b) if $x, y \in H$ and $x \neq y$, then there exist $U \in S(x)$ and $V \in S(y)$ for which $U \cap V = \emptyset$.*

Then from $|H| > \exp\exp\mathfrak{m}$ there follows the existence of a subset $D \subset H$ for which $|D| > \mathfrak{m}$ and $[S(x) \cap D] \setminus \{x\} = \emptyset$ for $x \in D$.

In ([1]) de Groot posed the problem: does there exist in every T_2-space of density greater than $\mathfrak{m}$ a discrete subspace of cardinality $> \mathfrak{m}$? In connection with this problem we proved the following result.

Theorem 2. *In every T_2-space of density $> \exp\mathfrak{m}$ there is a discrete subspace of cardinality $> \mathfrak{m}$.*

The following two theorems are devoted to the converse of theorem 2.

Let $s(R)$ be the density and $w(R)$ the weight of the space R. As is known, $w(R) \leq \exp s(R)$ for regular R. Hence one immediately obtains

Theorem 3. *If a regular space R contains a discrete subspace of cardinality $> \exp\mathfrak{m}$, then $s(R) > \mathfrak{m}$.*

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However, for Hausdorff spaces the analogous result does not hold.

Theorem 4. *Let* $\mathfrak{m}$ *be an arbitrary infinite cardinal number. Then there exists a Hausdorff space* R*, containing a discrete subspace of cardinality* $\exp\exp\mathfrak{m}$*, for which* $s(R) = \mathfrak{m}$*.*

The next theorem is interesting because it concerns discrete subspaces of T_1-spaces. As is known, the pseudocharacter $\psi(x, R)$ of a point x of a space R is the least cardinality of such a system of neighborhoods of the point x whose intersection is equal to $\{x\}$.

Theorem 5. *Let* R *be a* T_1*-space and let* $\mathfrak{m}$ *be an infinite cardinal number. Suppose that* $\psi(x, R) \leq \mathfrak{m}$ *at every point* $x \in R$ *and*

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* It is easy to see that from a positive solution of this problem of de Groot there would follow a positive solution of Suslin' s problem. However, the very latest investigations in axiomatic set theory show that Suslin's problem is independent of the usual axioms of set theory. Thus, in a certain sense theorem 2 can no longer be improved.

$\exp\mathfrak{m} < |R|$. Then R contains a discrete subspace of cardinality $> \mathfrak{m}$.

Corollary. Let R be a T_1-space satisfying the first axiom of countability. If $|R| > \exp\mathfrak{m}$, then R contains a discrete subspace of cardinality $> \mathfrak{m}$.

De Groot also posed the following problem: does every Hausdorff space R with $|R| = \aleph_\lambda$ contain a discrete subspace of cardinality $\aleph_\lambda$, if λ is a limit ordinal and if one assumes the generalized continuum hypothesis? In connection with this problem the following theorems have been obtained.

Theorem 6. Let R be a Hausdorff space and $|R| = \aleph_{\lambda+1}$, where λ is a limit number. If one assumes the generalized continuum hypothesis, then R contains a discrete subspace of cardinality $\aleph_\lambda$.

Theorem 7. Let R be a Hausdorff space and $|R| = \aleph_\lambda$, where λ is a limit number, cofinal with ω (that is, λ is representable as a countable sum of ordinal numbers smaller than λ). Then, if one assumes the generalized continuum hypothesis, R contains a discrete subspace of cardinality $\aleph_\lambda$.

§ 2. In ([4]) the following results are proved: a) If R is a Hausdorff space, $\mathfrak{m}$ is a regular cardinal number and $s(R) > \mathfrak{m}$, then R contains a subspace R' for which $s(R') = \mathfrak{m}$. b) An analogous result holds for singular $\mathfrak{m}$, however only under the assumption of the generalized continuum hypothesis. There the problem is also posed (Problem 4.2.1): can b) be freed from this hypothesis? The following theorems give partial solutions of this problem.

Theorem 8. Let R be a Hausdorff space and $s(R) > \aleph_\lambda$, where λ is a limit number cofinal with ω. Then R contains a subspace R' for which $s(R') = \aleph_\lambda$.

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Theorem 9. Let R be a linearly ordered topological space (where the basis of the topology is given by open intervals). If $\mathfrak{m}$ is an arbitrary cardinal number and $s(R) > \mathfrak{m}$, then R contains a subspace R' for which $s(R') = \mathfrak{m}$.

§ 3. The **character** $\chi(x, R)$ of a point x of a space R is the least cardinality of fundamental systems of neighborhoods of the point x.

Theorem 10. Let R be a Hausdorff space and $\mathfrak{m}$ an infinite cardinal number. If the set of those points $x \in R$ for which $\chi(x, R) \leq \mathfrak{m}$ has cardinality greater than $\exp \mathfrak{m}$, then R contains more than $\mathfrak{m}$ pairwise disjoint open subsets.

Let N be a countable discrete space and let βN be its Stone–Čech compactification. As E. Čech showed ([5]), for points $x \in \beta N \setminus N$,

$$\aleph_0 < \chi(x, \beta N) \leq 2^{\aleph_0}.$$

One of the authors posed the problem: how can one estimate more precisely the characters of the points $x \in \beta N \setminus N$? The first result in this direction is Theorem 11, whose proof is easily obtained by using Theorem 10.

Theorem 11. The set of those points $x \in \beta N \setminus N$ for which

$$\exp[\chi(x, \beta N)] < \exp \exp \aleph_0$$

has cardinality less than $\exp \exp \aleph_0$. (It is known that $|\beta N| = \exp \exp \aleph_0$.)

This result may be reformulated as follows: for almost all $x \in \beta N \setminus N$ the equality

$$\exp[\chi(x, \beta N)] = \exp \exp \aleph_0$$

holds.

It seems to us that, independently of the continuum hypothesis, it is impossible to characterize precisely the characters of the points $x \in \beta N \setminus N$.

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Note: Figure translations are in progress. See original paper for figures.

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