ON THE CAUCHY PROBLEM FOR GENERAL LINEAR EQUATIONS

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ON THE CAUCHY PROBLEM FOR GENERAL LINEAR EQUATIONS

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The present paper is devoted to establishing necessary and sufficient conditions for uniqueness of the solution of the Cauchy problem for equations of the form

$$P\left(\frac{\partial}{\partial x}, \frac{\partial}{\partial t}\right) u(x,t) = \sum_{0 \leqslant k \leqslant m} P_k\left(\frac{\partial}{\partial x}\right) \frac{\partial^k u(x,t)}{\partial t^k} = 0, \tag{1}$$

$$-\infty < x < \infty, \quad 0 \leqslant t < \infty,$$

where $P(s,\lambda)$ is an arbitrary polynomial with constant coefficients of degree N in s and degree m in λ, $P_m(s) \not\equiv 0$, under the initial conditions

$$\partial^j u(x,0)/\partial t^j = 0, \quad j = 0, \dots, m-1. \tag{2}$$

At the same time we shall consider only solutions of normal type in t, i.e., solutions of equation (1) which, together with all their derivatives entering equation (1), grow in t no faster than $\exp\{at\}$ with some $a > 0$.

The Cauchy problem for equations (and systems) of the form (1) was first studied in a special case by S. L. Sobolev (^{1}), and in the general formulation by S. A. Galpern (^{2,3}). Works (^{4,5}) are devoted to the same problem. In works ($^{1-4}$) the question of correct solvability of the Cauchy problem for the corresponding equations was studied, and in (^{5}) the uniqueness of the solution of this problem. In all these works restrictions were imposed on the form of the polynomial $P(s,\lambda)$ that were not dictated by the essence of the problem of uniqueness classes. We shall impose here no restrictions on the form of the polynomial $P(s,\lambda)$.

Let the roots $s_1(\lambda), \dots, s_N(\lambda)$ (not necessarily distinct) of the polynomial $P(s,\lambda)$ in a neighborhood of the infinitely distant point have the form (^{6})

$$s_j(\lambda) = a_j \lambda^{\gamma_j}(1 + o(1)), \quad j = 1, \dots, N, \quad a_j \neq 0, \quad o(1) \underset{|\lambda|\to\infty}{\longrightarrow} 0. \tag{3}$$

Denote

$$A = \min_{\{j:\, s_j(\lambda) \equiv \mathrm{const}\}} |\operatorname{Re} s_j(\lambda)|; \quad a = \min_{\{j:\, \gamma_j = 0\}} |\operatorname{Re} s_j(\lambda)|. \tag{4}$$

We divide the investigation into the following 5 cases, covering all possible cases: 1) $P_m(0) = 0$; $A > 0$, or roots of the form $s_j(\lambda) \equiv \mathrm{const}$ are absent; 2) $P_m(0) \neq 0$; $P_m(s) \not\equiv \mathrm{const}$; $a = 0$, $A > 0$, or roots of the form $s_j(\lambda) \equiv \mathrm{const}$ are absent; 3) $P_m(0) \neq 0$, $P_m(s) \not\equiv \mathrm{const}$; $a > 0$; 4) $A = 0$; 5) $P_m(s) \equiv \mathrm{const}$.

Theorem 1. *For uniqueness of the solution of the Cauchy problem (1)–(2) in cases 1) and 2) in the class of functions satisfying the estimate*

$$\left|\partial^k u(x,t)/\partial x^k\right| \leqslant C \exp\{\beta t + |x| H(x)\}, \tag{5}$$

$$k = 0, \ldots, N-1, \quad -\infty < x < \infty, \quad 0 \leqslant t < \infty, \quad \beta > 0,$$

where $H(x) > 0$ is an even continuous function, it is necessary and sufficient that

$$\inf H(x) = 0. \tag{6}$$

Proof. Denote by $y(x,\lambda)$ the Laplace transform (in t) of the solution $u(x,t)$ of problem (1)–(2). Then

$$P(\partial/\partial x, \lambda) y(x,\lambda) = 0, \tag{7}$$

and we must establish that condition (6) is necessary and sufficient in order that every solution $y(x,\lambda)$ of equation (7), analytic in some right half-plane and satisfying in this half-plane the estimate (following from (5)):

$$\left|\partial^k y(x,\lambda)/\partial x^k\right| \leq C_1 \exp\{|x| H(x)\}, \tag{8}$$

$k = 0, \ldots, N-1$, $-\infty < x < \infty$, be identically equal to zero.

Sufficiency. Let $y_1(x,\lambda), \ldots, y_N(x,\lambda)$ be a fundamental system of solutions of equation (7), and let $y(x,\lambda)$ be a solution of (7) satisfying condition (8). Representing $y(x,\lambda)$ in the form

$$y(x,\lambda) \equiv \sum_{1 \leq k \leq N} C_k(\lambda) y_k(x,\lambda), \tag{9}$$

we obtain

$$C_k(\lambda) = w^{-1}(x,\lambda) w_k(x,\lambda), \qquad k = 1, \dots, N, \tag{10}$$

where $w(x,\lambda)$ is the Wronskian of the system of functions $y_1(x,\lambda), \dots, y_N(x,\lambda)$, and $w_k(x,\lambda)$ is the determinant obtained from $w(x,\lambda)$ by replacing the k-th column by the column of functions $y(x,\lambda), \dots, y^{(N-1)}(x,\lambda)$. Taking into account that

$$w(x,\lambda) = w(0,\lambda) \exp\left\{ \sum_{1 \le j \le N} s_j(\lambda) x \right\}$$

and the estimate (8), from (10) we obtain

$$|C_k(\lambda)| \le C_2 (1 + |x|)^{M_1} |\lambda|^{M_2} \exp\{-\operatorname{Re} s_k(\lambda) x + |x| H(x)\}, \tag{11}$$

$-\infty < x < \infty$, $\operatorname{Re}\lambda \ge \sigma_0 > 0$; $M_1, M_2 > 0$.

From (11), taking (6) into account, we conclude that $C_k(\lambda) = 0$ for all λ, $\operatorname{Re}\lambda \ge \sigma_0$, with the exception of at most a finite number of rays. Indeed, if $\gamma_k > 0$, or $\gamma_k < 0$, or $\gamma_k = 0$ and $\operatorname{Re} a_k = 0$, then $\operatorname{Re} s_k(\lambda) \ne 0$ for any λ in some right half-plane, except, possibly, for a finite number of rays where $\operatorname{Re} s_k(\lambda) = 0$. For $\gamma_k = 0$ and $\operatorname{Re} a_k \ne 0$, the inequality $\operatorname{Re} s_k(\lambda) \ne 0$ is valid for all λ from some right half-plane. Fixing λ so that $\operatorname{Re} s_k(\lambda) \ne 0$, consider values of x such that $\operatorname{sign} x = \operatorname{sign} \operatorname{Re} s_k(\lambda)$, and choose from them a sequence x_n, $|x_n| \to \infty$, such that $H(x_n) \to 0$. Then as $n \to \infty$ the right-hand side of expression (11) tends to zero, and consequently $C_k(\lambda) = 0$. Thus the function (9) is equal to zero for $\operatorname{Re}\lambda \ge \sigma_0$, with the exception of at most a finite number of rays; consequently, $y(x,\lambda) \equiv 0$.

Necessity. In case 1), from the condition $P_m(0) = 0$ and the Newton diagram we conclude that there is a root $s_j(\lambda)$ with $\gamma_j < 0$. Let $\inf H(x) = C > 0$. Then the function $y(x,\lambda) = \exp\{s_j(\lambda)x\}$, not being identically zero, satisfies condition (8).

Analogously, in case 2) an example is provided by the function $\exp\{s_j(\lambda)x\}$, where

$$s_j(\lambda) = a_j + a_j' \lambda^{\gamma_j}(1 + o(1)), \quad \operatorname{Re} a_j = 0, \quad a_j' \ne 0, \quad \gamma_j < 0.$$

Theorem 2. *In case 3), the solution $u(x,t)$ of the Cauchy problem (1)–(2), satisfying the estimate*

$$\left|\partial^k u(x,t)/\partial x^k\right| \le C \exp\{\beta t + \alpha |x|\}, \tag{12}$$

$k = 0, \dots, N-1$, $-\infty < x < \infty$, $0 \le t < \infty$, $\beta > 0$, *is identically equal to zero if* $\alpha < a$. *If* $\alpha > a$, *then there exists a nontrivial solution of problem (1)–(2) satisfying estimate (12).*

Proof. As in the preceding theorem, it is enough to determine the existence of a nontrivial solution $y(x, \lambda)$ of equation (7), analytic in the right half-plane $\operatorname{Re}\lambda \ge \sigma_0 > 0$ and satisfying there the estimate

$$\left|\partial^k y(x,\lambda)/\partial x^k\right| \le C_1 \exp\{a|x|\}, \qquad k = 0, \dots, N-1, \quad -\infty < x < \infty. \tag{13}$$

Using (13), we arrive, analogously to (11), at the estimate $(k + 1, \dots, N)$

$$|C_k(\lambda)| \le C_2 (1 + |x|)^{M_1} |\lambda|^{M_2} \exp\{-\operatorname{Re} s_k(\lambda) x + \alpha |x|\} \tag{14}$$

for the coefficients of the expansion (9). Let $\alpha < a$. From the condition $P_m(0) \ne 0$ it follows that $\gamma_k \ge 0$, $k = 1, \dots, N$. Then from (3) and (4), for any $\varepsilon > 0$, for sufficiently large values of $|\lambda|$ we have $|\operatorname{Re} s_k(\lambda)| \ge a - \varepsilon$, $k = 1, \dots, N$. For $a - \varepsilon > \alpha$ the left-hand side of (14) tends to zero as $|x| \to \infty$ and $\operatorname{sign} x = \operatorname{sign} \operatorname{Re} s_k(\lambda)$. Consequently, $C_k(\lambda) \equiv 0$.

If $\alpha > a$, then the function $y(x, \lambda) = \exp\{s_k(\lambda) x\}$, where $|\operatorname{Re} a_k| = a$, $\gamma_k = 0$, is analytic for sufficiently large values of $|\lambda|$ and satisfies the estimate (13).

Remark. The class of functions satisfying the estimate (12) with $\alpha = a$ may turn out in case 3) to be both a uniqueness class for the solution of the problem (1)–(2) and a nonuniqueness class. This is indicated by the following examples: 1) $\partial^2 u/\partial x \partial t - \partial u/\partial t = u(x,t)$; 2) $\partial^2 u/\partial x \partial t - \partial u/\partial t = -u(x,t)$.

Theorem 3. *In case 4), for uniqueness of the solution of the Cauchy problem (1)–(2) in the class of functions satisfying the estimate*

$$|\partial^{ku}(x,t)/\partial x^k| \le \alpha(x) \exp\{\beta t\}, \tag{15}$$

$$k = 0, \dots, N-1, \quad -\infty < x < \infty, \quad 0 \le t < \infty, \quad \beta > 0,$$

$\alpha(x) > 0$ *an even function monotone for* $x > 0$, *it is necessary and sufficient that*

$$\lim_{x \to \infty} \alpha(x) = 0. \tag{16}$$

Proof. Suppose condition (16) is fulfilled. Analogously to (11) and (17), in the case under consideration, using (15), we obtain $(k = 1, \dots, N)$:

$$|C_k(\lambda)| \leq C_1(1+|x|)^{M_1}|\lambda|^{M_2}\alpha(x)\exp\{-\operatorname{Re} s_k(\lambda)x\} \tag{17}$$

$$-\infty < x < \infty, \quad M_1, M_2 > 0.$$

If $\operatorname{Re} s_k(\lambda) \neq 0$, then from (17) it is obvious that $C_k(\lambda) \equiv 0$. But $\operatorname{Re} s_k(\lambda) \neq 0$ for all λ in some right half-plane, with the exception of a finite number of rays, only if $s_k(\lambda) \not\equiv \text{const}$ or $s_k(\lambda) \equiv \text{const}$, but $\operatorname{Re} s_k(\lambda) \neq 0$. Thus the function $y(x,\lambda)$ is a linear combination of functions of the form $\exp\{s_k(\lambda)x\}$ and of products of such functions by powers of x, where $s_k(\lambda) \equiv \text{const}$, $\operatorname{Re} s_k = 0$. But then from (16) and the estimate $|y(x,\lambda)| \leq C\alpha(x)$, $-\infty < x < \infty$, it follows that $y(x,\lambda) \equiv 0$.

If condition (16) is not fulfilled, then equation (1) has a nontrivial solution of the form $u(x,t) = Ct^m \exp\{s_{kx}\}$, $\operatorname{Re} s_k = 0$, satisfying condition (15) and the initial condition (2).

In case 5), equation (1) is an equation of Kovalevskaya type. For equations of this type with reduced order $p_0 > 1$, in (7) a necessary and sufficient condition for uniqueness of the problem (1)–(2) was established, but in a form inconvenient for verification. We shall give here a more convenient criterion.

Theorem 4. *Let $H(x) > 0$ be an even function increasing for $x > 0$. If the reduced order of equation (1) is $p_0 > 1$, then for uniqueness of the solution of the problem (1)–(2) in the class of functions satisfying the estimate*

$$|\partial^{k u}(x,t)/\partial x^k| \leq C\exp\left\{\beta t + \left|\int_0^x H(t)\,dt\right|\right\}, \tag{18}$$

$$k = 0,\dots,N-1, \quad -\infty < x < \infty, \quad 0 \leq t < \infty, \quad \beta > 0,$$

it is necessary and sufficient that

$$\int^{\infty} [H(x)]^{1-p_0}\,dx = \infty. \tag{19}$$

The proof follows the same scheme as above. Similarly to (11), (14), and (17), we obtain ($k = 1,\dots,N$):

$$|C_k(\lambda)| \leq C_1(1+|x|)^{M_1}|\lambda|^{M_2}\exp\left\{-\operatorname{Re} s_k(\lambda)x + \left|\int_0^x H(t)\,dt\right|\right\}, \qquad M_1,\ M_2 > 0. \tag{20}$$

The functions $C_k(\lambda)$, $k = 1,\dots,N$, are analytic for $\operatorname{Re}\lambda \geq \sigma_0 > 0$ (σ_0 sufficiently large). Under the assumption (19) we shall show the strong decrease of each of the functions $C_k(\lambda)$ along one of the half-lines $\lambda = \sigma_1 + i\tau$, $\sigma_1 > \sigma_0$, $0 < \tau < \infty$, or $-\infty < \tau < 0$, or $\lambda = \sigma(1 + Ai)$, $0 < \sigma < \infty$, from which we shall be able to conclude that $C_k(\lambda) \equiv 0$, $\operatorname{Re}\lambda \geq \sigma_0$. Since $\min \gamma_k = p_0^{-1}$ (7), then for $\gamma_k = p_0^{-1} < 1$ on one of the half-lines $\lambda = \sigma_1 + i\tau$, $\tau \geq \tau_0$ or $\tau \leq -\tau_0$, $|\operatorname{Res}_k(\lambda)| \geq C_k|\tau|^{p_0^{-1}}$. Substituting this estimate into (20), choosing the sign of x in the required way, and putting $|x| = x(|\tau|)$, where $H(x(|\tau|)) \equiv \frac{1}{2}C_k|\tau|^{p_0^{-1}}$, we obtain from (20)

$$|C_k(\sigma_1 + i\tau)| \leq C_2 \exp\{-C'|\tau|^{p_0^{-1}} x(|\tau|)\},$$

whence, using the definition of $x(|\tau|)$ and condition (19), we conclude that

$$\int^{\infty} |\tau|^{-2} \ln |C_k(\sigma_1 + i\tau)| \, d|\tau| = -\infty.$$

It follows from this, by (8), that $C_k(\lambda) \equiv 0$. For $\gamma_k > p_0^{-1}$ an analogous conclusion is obtained by considering $C_k(\lambda)$ for $\lambda = \sigma(1 + Ai)$, $\sigma > 0$, A sufficiently large.

If condition (19) is not fulfilled, then an example of a nontrivial solution $u(x,t)$ of problem (1)–(2), satisfying condition (18), is given by the inverse Laplace transform of the function $y(x,\lambda) = C(\lambda)\exp\{s_j(\lambda)x\}$, $\gamma_j = p_0^{-1}$, where $C(\lambda)$ is a function analytic in the right half-plane such that

$$|C(\lambda)| \leq C_1 \exp\{-C_2|\lambda|^{p_0^{-1}} x(|\lambda|)\},$$

with $H(x(|\lambda|)) \equiv C_3|\lambda|^{p_0^{-1}}$, $C_3 < C_2$. The existence of such a $C(\lambda) \neq 0$ follows from the Carleman-Ostrovsky criterion, since

$$\int^{\infty} |\lambda|^{p_0^{-1}-2} x(|\lambda|) \, d|\lambda| < \infty.$$

Remark. If $p_0 \leq 1$, then problem (1)–(2) has a unique solution in the class of functions (18) for any increasing function $H(x) > 0$, since condition (19) will always be fulfilled; the proof is analogous to the first part of the proof of Theorem 4.

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Note: Figure translations are in progress. See original paper for figures.

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