ON THE TOPOLOGICAL CLASSIFICATION OF NONSINGULAR SURFACES OF THE FOURTH ORDER

MATHEMATICS

1967

SovietRxiv

Abstract

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UDC 513.6(07)

MATHEMATICS

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ON THE TOPOLOGICAL CLASSIFICATION OF NONSINGULAR SURFACES OF THE FOURTH ORDER

(Presented by Academician I. G. Petrovsky, 16 IX 1966)

Definition 1. A **piece** of a real algebraic surface F in the space $R^3(x, y, z, u)$ ($R^3(x, y, z, u)$ is real projective three-dimensional space) will mean any component of the surface F in the space R^3.

Definition 2. Two real algebraic surfaces F_1 and F_2 without real singular points, isotopic in R^3, will be called **surfaces of the same type**.

Definition 3. The **genus of a real algebraic surface** F **without real singular points** (see ([5])) is the maximum number of one-dimensional intersecting closed lines that can be drawn on all pieces of the surface F without causing this surface to split into a larger number of pieces.

The **genus of a piece of the surface** F is defined analogously. Obviously, with this definition the genus of the surface F is equal to the sum of the genera of its constituent pieces.

Definition 4. The number $r_l = g_l + 1$, where g_l is the genus of the piece $K_l \subset F$, is called the **rank of the piece** K_l, and the number $r = \Sigma r_l$ (the sum is taken over all pieces of the surface F) is called the **rank of the surface** F (see ([4]), pp. 449–453).

In what follows, for geometric constructions we shall consider the usual model of the space $R^3(x, y, z, u)$ in the form of a unit ball with diametrically opposite points of the bounding sphere S identified, which represents the plane $u = 0$.

Let $Q \subset R^3$ be a sphere concentric with the sphere S, and let its radius be $\rho < 1$. The center of the spheres S and Q is the point $(0, 0, 0, 1)$. Through the center of the sphere Q draw a plane π. Cut out on the sphere Q two "films" Q_1 and Q_2, i.e. two pieces, each of which is homeomorphic to a disk. To the two "holes" of the set $Q \setminus (Q_1 \cup Q_2)$ attach a handle B_1, homeomorphic to a cylinder, in such a way that the handle is not knotted. In this case there are two possibilities:

a) by a continuous deformation of the handle in R^3 it can be placed so that it does not intersect the plane $u = 0$ at any real point. Such a handle we shall call "finite." Moreover, topologically one may assume that this handle does not intersect the plane π either.

b) The handle is not "finite." Then by a continuous deformation of the handle in R^3 it can be placed so that it intersects the plane $u = 0$ in exactly one oval and does not intersect the plane π. Such a handle we shall call "infinite." Denote the resulting surface $[Q \setminus (Q_1 \cup Q_2)] \cup B_1$ by F_1.

We shall carry out the further construction inductively. Suppose that a surface F_p of genus p has already been constructed. Cut out on it two "films" Q_{2p+1} and Q_{2p+2}, each of which is homeomorphic to a disk. To the surface $F_p \setminus (K_{2p+1} \cup Q_{2p+2})$ attach the previously cut-out "films" $Q_1, Q_2, \dots, Q_{2p-1}, Q_{2p}$. To the two "holes" of the resulting set

$$F'_p = [F_p \setminus (Q_{2p+1} \cup Q_{2p+2})] \cup \left(\bigcup_{j=1}^{2p} Q_j \right)$$

attach ···

Table 1

$11R_0^0$	$R_1^0+10R_0^0$	$R_2^0+10R_0^0$	$R_3^0+7R_0^0$	$R_4^0+6R_0^0$	$R_5^0+5R_0^0$	$R_6^0+5R_0^0$	$R_7^0+3R_0^0$	$R_8^0+2R_0^0$	$R_9^0+2R_0^0$	$R_{10}^0+2R_0^0$	$2R_1^0+R_0^0$
$10R_0^0$	$R_1^0+9R_0^0$	$R_2^0+9R_0^0$	$R_3^0+6R_0^0$	$R_4^0+5R_0^0$	$R_5^0+4R_0^0$	$R_6^0+4R_0^0$	$R_7^0+2R_0^0$	$R_8^0+R_0^0$	$R_9^0+R_0^0$	$R_{10}^0+R_0^0$	$2R_1^0$
$9R_0^0$	$R_1^0+8R_0^0$	$R_2^0+8R_0^0$	$R_3^0+5R_0^0$	$R_4^0+4R_0^0$	$R_5^0+3R_0^0$	$R_6^0+3R_0^0$	$R_7^0+R_0^0$	R_8^0	R_9^0	R_{10}^0	$(R_0^0;\ R_0^0)$
$8R_0^0$	$R_1^0+7R_0^0$	$R_2^0+7R_0^0$	$R_3^0+4R_0^0$	$R_4^0+3R_0^0$	$R_5^0+2R_0^0$	$R_6^0+2R_0^0$	R_7^0				
$7R_0^0$	$R_1^0+6R_0^0$	$R_2^0+6R_0^0$	$R_3^0+3R_0^0$	$R_4^0+2R_0^0$	$R_5^0+R_0^0$	$R_6^0+R_0^0$					
$6R_0^0$	$R_1^0+5R_0^0$	$R_2^0+5R_0^0$	$R_3^0+2R_0^0$	$R_4^0+R_0^0$	R_5^0	R_6^0					
$5R_0^0$	$R_1^0+4R_0^0$	$R_2^0+4R_0^0$	$R_3^0+R_0^0$	R_4^0							
$4R_0^0$	$R_1^0+3R_0^0$	$R_2^0+3R_0^0$	R_3^0								
$3R_0^0$	$R_1^0+2R_0^0$	$R_2^0+2R_0^0$									
$2R_0^0$	$R_1^0+R_0^0$	$R_2^0+R_0^0$									
R_0^0	R_1^0	R_2^0									
0											

glue, without intersection with the set $\widetilde{F}_p$ and the plane π, a nonsingular handle B_{p+1}, homeomorphic to a cylinder. The surface

$$(\widetilde{F}_p \cup B_{p+1}) \setminus \left(\bigcup_{j=1}^{2p} Q_j \right)$$

of genus $(p+1)$ we shall call the surface F_{p+1}.

Definition 5. If the surface F_p of genus p, obtained in the manner indicated above, has k "infinite" and $p-k$ "finite" handles, and none of the handles of the surface F_p intersects the plane π, then we shall denote such a surface by R_p^k. A surface in R^3 isotopic to a surface R_p^k will be called a **piece of type** R_p^k. A piece of type R_0^0 is called an **oval**.

Definition 6. We shall say that a surface F has the following types:

a) $R_q^0 + kR_0^0$ $(q \geqslant 1)$, if it is isotopic in R^3 to a surface consisting of one piece of type R_q^0 and k pieces of type R_0^0 (i.e., ovals), moreover

Table 2

$R_1^1+10R_0^0$	$R_2^1+10R_0^0$	$R_3^1+7R_0^0$	$R_4^1+6R_0^0$	$R_5^1+5R_0^0$	$R_6^1+5R_0^0$	$R_7^1+3R_0^0$	$R_8^1+2R_0^0$	$R_9^1+2R_0^0$	$R_{10}^1+2R_0^0$	$R_1^1+R_1^0+R_0^0$
$R_1^1+9R_0^0$	$R_2^1+9R_0^0$	$R_3^1+6R_0^0$	$R_4^1+5R_0^0$	$R_5^1+4R_0^0$	$R_6^1+4R_0^0$	$R_7^1+2R_0^0$	$R_8^1+R_0^0$	$R_9^1+R_0^0$	$R_{10}^1+R_0^0$	$R_1^1+R_1^0$
$R_1^1+8R_0^0$	$R_2^1+8R_0^0$	$R_3^1+5R_0^0$	$R_4^1+4R_0^0$	$R_5^1+3R_0^0$	$R_6^1+3R_0^0$	$R_7^1+R_0^0$	R_8^1	R_9^1	R_{10}^1	$2R_1^1$
$R_1^1+7R_0^0$	$R_2^1+7R_0^0$	$R_3^1+4R_0^0$	$R_4^1+3R_0^0$	$R_5^1+2R_0^0$	$R_6^1+2R_0^0$	R_7^1				
$R_1^1+6R_0^0$	$R_2^1+6R_0^0$	$R_3^1+3R_0^0$	$R_4^1+2R_0^0$	$R_5^1+R_0^0$	$R_6^1+R_0^0$					
$R_1^1+5R_0^0$	$R_2^1+5R_0^0$	$R_3^1+2R_0^0$	$R_4^1+R_0^0$	R_5^1	R_6^1					
$R_1^1+4R_0^0$	$R_2^1+4R_0^0$	$R_3^1+R_0^0$	R_4^1							
$R_1^1+3R_0^0$	$R_2^1+3R_0^0$	R_3^1								
$R_1^1+2R_0^0$	$R_2^1+2R_0^0$									
$R_1^1+R_0^0$	$R_2^1+R_0^0$									
R_1^1	R_2^1									

all ovals are situated outside one another and in the exterior region of the piece of type R_q^0.*

b) lR_0^0, if it is isotopic in R^3 to a surface consisting of l ovals situated outside one another.

c) $R_p^1 + kR_0^0$ ($p \geqslant 1$), if it is isotopic in R^3 to a surface consisting of one piece of type R_p^1 and k ovals, all the ovals being situated outside one another and in one of the regions into which space R^3 is divided by the piece of type R_p^1 ($p \geqslant 1$).**

d) $R_1^1 + R_1^0 + pR_0^0$ ($p \geqslant 0$), if it is isotopic in R^3 to a surface consisting of one piece of type R_1^1, one piece of type R_1^0, and p ovals,

* Obviously, a piece of type R_q^0 divides space R^3 into two regions—exterior and interior.

** We note that a piece of type R_p^1 divides space R^3 into two regions, and the type of the surface does not depend on which region contains all the ovals.

moreover, the piece of type R_1^0 and all the ovals are situated outside one another and in one of the regions into which the space R^3 is divided by the piece of type R_1^1.

d) $(R_0^0; R_0^0)$, if it is isotopic in R^3 to a surface consisting of two ovals, one of them being situated inside the other.

e) $2R_1^0$, if it is isotopic in R^3 to a surface consisting of two pieces of type R_1^0 situated outside one another and not linked.

f) $2R_1^1$, if it is isotopic in R^3 to a surface consisting of two pieces of type R_1^1.

In all cases the pieces have no common points.

Theorem 1*. *A real algebraic surface of the 4th order without real singular points is a surface of one of the types listed in Tables 1 and 2.*

For the proof of Theorem 1, a theory is constructed concerning the relation between real algebraic surfaces of the 4th order which have one real double point, and real algebraic curves of the 6th order without real singular points. This theory makes it possible, using the complete topological classification of nonsingular real algebraic curves of the 6th order carried out in ([6]) (see also ([7]), p. 482), to prove Theorem 1. Moreover, the indicated theory makes it possible to draw a conclusion about the existence of a surface of the 4th order of a given type if the existence is known of a curve of the 6th order possessing the prescribed properties. The construction of such curves proves the following theorems.

Theorem 2.** *Surfaces of the 4th order of those types which are located below the broken line in Tables 1 and 2 exist.*

Theorem 3. *At least one of the surfaces of the types $R_p^1 + kR_0^0$ and $R_p^0 + kR_0^0$ exists for the cases $p = 2,\ 2 \leq k \leq 7$ and $p = 3,\ k = 3, 4$ (in Tables 1 and 2 they are separated by a dotted line).*

Thus, for example, the surface of the 4th order given by the equation

$$F(x,y,z,u) \equiv (4x^2+4y^2-z^2)u^2 + 2\sqrt{15}\,(16x^2y - yz^2)u +$$
$$+[60x^4 - 28z^4 - 176x^3z + 528xy^2z + 540x^2y^2 + 144x^2z^2 + 129y^2z^2 +$$
$$+\varepsilon(x^2+y^2+z^2)^2] - \delta u^4 = 0,$$

where $\varepsilon > 0$ and $\delta = \delta(\varepsilon) > 0$ are sufficiently small, has type $R_1^1 + 9R_0^0$.

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named after N. I. Lobachevsky

Received
21 VII 1966

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* K. Rohn dealt with analogous questions. In works ([1]-[3]), K. Rohn: 1) established a relation between real algebraic surfaces of the 4th order which have at least one real double point and real algebraic curves of the 6th order; 2) on the basis of the indicated relation proved that a real algebraic surface of the 4th order cannot have more than 12 pieces; 3) proved that surfaces of the types $12R_0^0$ and $11R_0^0$ (our notation) do not exist; 4) constructed a surface of type $10R_0^0$; 5) at the end of work ([1]) attempted to give a definition of the type of a surface and to list the possible types of surfaces of the 4th order without real singular points; however, there are no proofs of the assertions made in this connection either in work ([1]) or in K. Rohn' s other works.

** D. Hilbert (([4]), pp. 449–453) gave the construction of a surface of the 4th order which, in our notation, has type $R_{10}^1 + R_0^0$. Obviously, its rank is $r = 12$.

Note: Figure translations are in progress. See original paper for figures.