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Abstract

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K-CONTINUABILITY OF POLYNOMIALS

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In the present paper a simple algebraic method is considered for solving the problem of K-continuability of polynomials.

The problem of continuability of polynomials is as follows. Given a polynomial

$$P(x) = 1 + a_1 x + a_2 x^2 + \dots + a_{nx}^n;$$

we form the polynomial

$$P_1(x) = P(x) + a_{n+1} x^{n+1} + \dots + a_{mx}^m.$$

One asks under what conditions the roots of the polynomial $P_1(x)$, with a suitable choice of m and of the coefficients $a_{n+1}, \dots, a_m$, can be located on a prescribed set of points of the plane of the complex variable. It was already proved long ago by me that the problem is always possible if the set of points is taken to be the circle $|z| = 1$ (K-continuability of polynomials).

Making the transformation $x = 1/z$, we obtain

$$P_1(1/z) z^m = (z^n + a_1 z^{n-1} + \dots + a_n) z^{m-n} + a_{n+1} z^{m-n-1} + \dots + a_m.$$

In this form it is more convenient to consider the problem, since Newton' s formulas for computing sums of like powers of the roots of a polynomial are usually written for the case when the coefficient of the highest power is equal to one.

In my preceding paper the following solution of the problem of K-continuability of polynomials was given (see ([1])):

$$s_k = \sum_{j=1}^{n} N_j (e^{ik\alpha_j} + e^{-ik\alpha_j})(\varepsilon_{1j}^k + \varepsilon_{2j}^k + \dots + \varepsilon_{jj}^k) e^{ik\psi_j}, \qquad k = 1, 2, \dots, n,$$

where s_k are Newton sums; $\varepsilon_{1j}, \varepsilon_{2j}, \dots, \varepsilon_{jj}$ are the roots of the equation $z^j = 1$; N_j are positive integers.

The roots of the continued polynomial are seen in the first line

$$s_1 = N_1(e^{i\alpha_1} + e^{-i\alpha_1})e^{i\psi_1} + N_2(e^{i\alpha_2} + e^{-i\alpha_2})(\varepsilon_{12} + \varepsilon_{22})e^{i\psi_2} + \dots$$

$$\dots + N_n(e^{i\alpha_n} + e^{-i\alpha_n})(\varepsilon_{1n} + \varepsilon_{2n} + \dots + \varepsilon_{nn})e^{i\psi_n}.$$

The first group of roots has multiplier N_1, the second has N_2, and so on.

Let us form the continued polynomial. For this we form the polynomials corresponding to the first group of roots, then to the second group of roots, and so on. For the first group of roots

$$\{[z - e^{i(\alpha_1+\psi_1)}][z - e^{i(-\alpha_1+\psi_1)}]\}^{N_1} = [z^2 - (e^{i\alpha_1} + e^{-i\alpha_1})e^{i\psi_1} + e^{2i\psi_1}]^{N_1}.$$

By Euler' s formula

$$\rho_1/N_1 = e^{i\alpha_1} + e^{-i\alpha_1} = 2\cos\alpha_1 \geq 0, \qquad s_1 = \rho_1 e^{i\varphi_1}$$

(see ([1])), denoting $-2\cos\alpha_1 e^{i\psi_1} = b_1$, we obtain

$$[z^2 + b_1 z + e^{i\cdot 2\psi_1}]^{N_1} = z^{2N_1} + N_1 b_1 z^{2N_1-1} + cz^{2N_1-2} + \dots,$$

where $|b_1| \leq 2$, $\psi_1 = \arg b_1 + \pi$ (since $\cos\alpha_1 \geq 0$); it is easy to see that, when these conditions are satisfied, the roots of the quadratic polynomial lie on the unit circle.

Next we form the polynomial corresponding to the second group of roots. We obtain

$$[z^4 + b_2 z^2 + e^{i4\psi_2}]^{N_2} = z^{4N_2} + N_2 b_2 z^{4N_2-2} + \cdots,$$

where $b_2 = -(e^{2i\alpha_2} + e^{-2i\alpha_2})e^{2i\psi_2}$, $|b_2| \leq 2$, $2\psi_2 = \arg b_2 + \pi$.

For the last group of roots we obtain the polynomial

$$[z^{2n} + b_{nz}^n + e^{i\cdot 2n\psi_n}]^{N_n} = z^{2nN_n} + N_{nb_{nz}}^{2nN_n-n} + \cdots,$$

where $b_n = -(e^{in\alpha_n} + e^{-in\alpha_n})e^{in\psi_n}$, $|b_n| \leq 2$, $n\psi_n = \arg b_n + \pi$.

Consequently, the extended polynomial is composed in the form

$$[z^2+b_1z+e^{2i\psi_1}]^{N_1}[z^4+b_2z^2+e^{i\cdot 4\psi_2}]^{N_2}\cdots[z^{2n}+b_{nz}^n+e^{i\cdot 2n\psi_n}]^{N_n}$$
$$=(z^n+a_1z^{n-1}+a_2z^{n-2}+\cdots+a_n)z^{m-n}+a_{n+1}z^{m-n-1}+\cdots+a_m,$$

where $m=2N_1+4N_2+\cdots+2nN_n$; the numbers $b_1,\psi_1,b_2,\psi_2,\ldots,b_n,\psi_n$ are found successively.

We first observe that multiplying the leading terms of the factors gives the leading term of the product. If in the first factor one takes the second term $N_1b_1z^{2N_1-1}$, and in all the other factors the leading terms, then we obtain the product $N_1b_1z^{2N_1+4N_2+\cdots+2nN_n-1}=N_1b_1z^{m-1}$, which represents the second term of the extended polynomial. Consequently, $N_1b_1=a_1$, $b_1=a_1/N_1$; N_1 must be taken so large that $|b_1|\le 2$. As we see, this can be done in infinitely many ways. Having found b_1, we determine the argument of b_1, which we denote by $\psi_1-\pi$, and the first factor is determined. After this we proceed to finding the second factor. For this we note that the next term of the extended polynomial is obtained if in the second factor we take the second summand $N_2b_2z^{4N_2-2}$, and in the remaining factors the first terms. The product obtained is

$$N_2b_2z^{2N_1+4N_2+\cdots+2nN_n-2}=N_2b_2z^{m-2}.$$

But, in addition, one can take in the second and subsequent factors the leading terms, and in the first factor the term cz^{2N_1-2}. The product obtained is

$$cz^{2N_1+4N_2+\cdots+2nN_n-2}=cz^{m-2}.$$

Thus,

$$N_2b_2+c=a_2,$$

whence we find $b_2=(a_2-c)/N_2$, where N_2 is sufficiently large. Having found b_2, we determine ψ_2, and so on.

Let us consider, as an example, the polynomial

$$z^2+\tfrac{1}{2}z+2.$$

We know that a real polynomial extends real-ly. Write

$$[z^2+b_1z+1]^{N_1}[z^4+b_2z^2+1]^{N_2}=(z^2+\tfrac{1}{2}z+2)z^{m-2}+\cdots,$$

where $m=2N_1+4N_2$.

First, $N_1b_1=\frac{1}{2}$; we may take $N_1=1$, $b_1=\frac{1}{2}$. We shall have

$$[z^2+\tfrac{1}{2}z+1][z^4+b_1z^2+1]^{N_2}=(z^2+\tfrac{1}{2}z+2)z^{m-2}+\cdots,$$
$$m=2+4N_2.$$

Next, $N_2b_2+1=2$, $N_2b_2=1$; we may take $N_2=1$, $b_2=1$, consequently,

$$(z^2+\tfrac{1}{2}z+1)(z^4+z^2+1)=(z^2+\tfrac{1}{2}z+2)z^4+\tfrac{1}{2}z^3+2z^2+\tfrac{1}{2}z+1,$$

and the problem is solved.

We note that the roots of the extended polynomial are expressible in radicals. This simplest method for solving the problem of K-extendability of polynomials was found by me by conjecture. It is only one of an infinite number of methods. The general principles for solving the problem were developed by me in other works and were used for further generalizations. Precisely because the problem has an infinite number of solutions, various generalizations have proved possible.

In my previous works it was proved that part of the points of the circle can be removed without violating the generality of the theorem (see [2]).

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REFERENCES

[1] L. I. Gavrilov, DAN, **135**, No. 3 (1960). [2] *Mathematics in the USSR over 40 Years*, 2, Moscow, 1959.

Note: Figure translations are in progress. See original paper for figures.