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Abstract

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CYBERNETICS
AND CONTROL THEORY

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ON ASYMPTOTIC INTEGRATION OF EQUATIONS OF A CONTROLLED PROCESS

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1. A controlled process in the linear formulation is usually described by a system of equations that can be represented in the form

$$A(t)\frac{dx}{dt} = B(t)x + \sum_{\nu=1}^{l} a_\nu(t)\delta_\nu,$$

$$\delta_\nu = \int_{-\infty}^{t} g_\nu(t-t', t')v_\nu(t')\,dt', \qquad v_\nu = b_\nu(t)x(t), \tag{1}$$

where x is a column matrix composed of the coordinated values of the controlled quantities; δ_ν, v_ν, g_ν are, respectively, the output signals (control functions), input signals, and impulse transition functions of the control system; b_ν are row matrices determining the law by which the input signal is formed; a_ν are column matrices; A, B are square matrices of order n.

It is assumed that the elements of the matrices A, B, a_ν, b_ν are slowly varying functions. The g_ν are also slowly varying as functions of the second argument. All these functions have a sufficient number of derivatives with respect to their arguments. In addition, it is assumed that the column matrices a_ν, which determine the action of the regulator on the controlled process, are "small."

The solution of a system similar to (1), in the case of simple eigenvalues of the matrix $U = A^{-1}B$, was given in (^{1}). In the general case, when among the eigenvalues of the matrix U there are also equal and intersecting ones, a system of the form (1), as shown in (^{2}), can be transformed into a system of differential equations consisting of several independent subsystems of lower order.

Below it is shown that if all eigenvalues of a given isolated group, among which there may also be equal ones, correspond only to linear elementary divisors of the characteristic matrix $\lambda E_n - U$, then under a certain condition a complete splitting of the corresponding subsystem of the transformed system of equations can be realized. As in the cited works, we use the method of introducing slow time, proposed by N. M. Krylov and N. N. Bogolyubov (^{3}).

2. Instead of the system (1), we shall henceforth consider the integro-differential equation

$$A(\tau)\frac{dx}{dt} = B(\tau)x + \varepsilon \sum_{\nu} a_\nu(\tau) \int_{-\infty}^{t} g_\nu(t-t',\tau')b_\nu(\tau')x(t')\,dt' \qquad (\tau = \varepsilon t), \tag{2}$$

which for $\varepsilon = 1$ is equivalent to the system (1).

Divide the eigenvalues of the matrix U into p groups $\lambda_1^{(\sigma)}, \dots, \lambda_{k_\sigma}^{(\sigma)}$

$$\left(\sigma = 1, \dots, p; \quad \sum_{\sigma=1}^{p} k_\sigma = n\right)$$

under the condition that on the segment $0 \le \tau \le L$

$$\left|\lambda_i^{(\sigma)}(\tau) - \lambda_j^{(s)}(\tau)\right| \ge c > 0 \quad (s \ne \sigma;\ i = 1, \dots, k_\sigma;\ j = 1, \dots, k_s). \tag{3}$$

Then square matrices can be constructed

$$K = (K_1 \dots K_p), \qquad \Lambda = \begin{bmatrix} \Lambda_1 & 0 & \\ & \ddots & \\ 0 & & \Lambda_p \end{bmatrix}, \qquad M = \begin{bmatrix} M_1 \\ \vdots \\ M_p \end{bmatrix}$$

with submatrices $K_\sigma, \Lambda_\sigma, M_\sigma$ of types respectively $n \times k_\sigma$, $k_\sigma \times k_\sigma$, $k_\sigma \times n$, satisfying the relations

$$U(\tau) = K(\tau)\Lambda(\tau)M(\tau) = \sum_{\sigma=1}^{p} K_\sigma(\tau)\Lambda_\sigma(\tau)M_\sigma(\tau), \tag{4}$$

$$M(\tau)K(\tau) = K(\tau)M(\tau) = E_n, \tag{5}$$

$$M_s(\tau)K_\sigma(\tau) = \begin{cases} E_{k_\sigma}, & (s = \sigma), \\ 0, & (s \ne \sigma), \end{cases} \tag{6}$$

and differentiable on $[0, L]$ with respect to τ as many times as the matrix U is differentiable (see $(4, 5)$).

Let group σ consist of k_σ identically equal eigenvalues with common value λ_σ, and let each of them correspond to a linear elementary divisor of the matrix $\lambda E_n - U$. We shall show that if, in this case, the matrix

$$\Lambda_\sigma^{[1]} = -M_\sigma \left[\frac{dK_\sigma}{d\tau} - A^{-1} \sum_\nu a_\nu b_\nu R^{(\nu)}(\lambda_\sigma, \tau) K_\sigma \right], \tag{7}$$

where

$$R^{(\nu)}(\lambda, t) = \int_0^\infty g_\nu(s, t) e^{-\lambda s}\, ds \tag{8}$$

—the transfer function of the control system with parameters "frozen" at the given instant of time t—has on $[0, L]$ only distinct eigenvalues, then the formal solution of equation (2) corresponding to the group σ of eigenvalues of the matrix U can be represented by the equalities

$$x_\sigma = \widetilde{K}_\sigma(\tau, \varepsilon) \widetilde{G}_\sigma(\tau_1, \varepsilon) y_\sigma \qquad (\tau_1 = \varepsilon\tau), \tag{9}$$

$$dy_\sigma/dt = \widetilde{W}_\sigma(\tau, \varepsilon) y_\sigma, \tag{10}$$

where $\widetilde{K}_\sigma, \widetilde{G}_\sigma, \widetilde{W}_\sigma$ are matrices of types respectively $n \times k_\sigma$, $k_\sigma \times k_\sigma$, $k_\sigma \times k_\sigma$, representable by formal series

$$\widetilde{K}_\sigma(\tau, \varepsilon) = \sum_{k=0}^\infty \varepsilon^k K_\sigma^{[k]}(\tau), \qquad \widetilde{G}_\sigma(\tau_1, \varepsilon) = \sum_{k=0}^\infty \varepsilon^k G_\sigma^{[k]}(\tau_1),$$

$$\widetilde{W}_\sigma(\tau, \varepsilon) = \lambda_\sigma E_{k_\sigma} + \varepsilon \sum_{k=0}^\infty \varepsilon^k W_\sigma^{[k]}(\tau), \tag{11}$$

and all $W_\sigma^{[k]}$ are diagonal matrices.

The fundamental matrix of solutions of equation (10) has the form

$$Y_\sigma = \exp \theta_\sigma(t, \varepsilon), \tag{12}$$

where θ_σ is a diagonal matrix of order k_σ such that

$$d\theta_\sigma/dt = \widetilde{W}_\sigma(\tau, \varepsilon). \tag{13}$$

Substitute the vector x_σ, defined by the equalities (9) and (10), into (2). Then, using the solution (12) of equation (10) and putting

$$\widetilde{G}_\sigma(\tau_1,\varepsilon)\widetilde{W}_\sigma(\tau,\varepsilon)+\varepsilon^2\frac{d\widetilde{G}_\sigma}{d\tau_1}(\tau_1,\varepsilon)=\widetilde{\Lambda}_\sigma(\tau,\varepsilon)\widetilde{G}_\sigma(\tau_1,\varepsilon), \tag{14}$$

we obtain

$$\left\{A(\tau)\left[\widetilde{K}_\sigma(\tau,\varepsilon)\widetilde{\Lambda}_\sigma(\tau,\varepsilon)+\varepsilon\frac{d\widetilde{K}_\sigma}{d\tau}(\tau,\varepsilon)\right]-B(\tau)\widetilde{K}_\sigma(\tau,\varepsilon)\right\}\widetilde{G}_\sigma(\tau_1,\varepsilon)=\varepsilon\sum_\nu a_\nu(\tau)I_\sigma^{(\nu)}, \tag{15}$$

where

$$I_\sigma^{(\nu)}=\int_{-\infty}^{t} g_\nu(t-t',\tau')\,b_\nu(\tau')\,\widetilde{K}_\sigma(\tau',\varepsilon)\,\widetilde{G}_\sigma(\tau_1,\varepsilon)\exp[\theta_\sigma(t',\varepsilon)-\theta_\sigma(t,\varepsilon)]\,dt'. \tag{16}$$

The integral (16) can be transformed to the form

$$\begin{aligned}I_\sigma^{(\nu)}&=b_\nu(\tau)\widetilde{K}_\sigma(\tau,\varepsilon)\widetilde{G}_\sigma(\tau_1,\varepsilon)R_{00}^{(\nu)}[\widetilde{W}_\sigma(\tau,\varepsilon),\tau]\\&\quad+\varepsilon\left\{\left[b_\nu(\tau)\frac{d\widetilde{K}_\sigma(\tau,\varepsilon)}{d\tau}\widetilde{G}_\sigma(\tau_1,\varepsilon)+\frac{db_\nu(\tau)}{d\tau}\widetilde{K}_\sigma(\tau,\varepsilon)\widetilde{G}_\sigma(\tau_1,\varepsilon)\right]R_{10}^{(\nu)}[\widetilde{W}_\sigma(\tau,\varepsilon),\tau]\right.\\&\qquad+b_\nu(\tau)\widetilde{K}_\sigma(\tau,\varepsilon)\widetilde{G}_\sigma(\tau_1,\varepsilon)R_{11}^{(\nu)}[\widetilde{W}_\sigma(\tau,\varepsilon),\tau]\\&\qquad\left.+\frac{1}{2}b_\nu(\tau)\widetilde{K}_\sigma(\tau,\varepsilon)\widetilde{G}_\sigma(\tau_1,\varepsilon)\frac{d\widetilde{W}_\sigma(\tau,\varepsilon)}{d\tau}R_{20}^{(\nu)}[\widetilde{W}_\sigma(\tau,\varepsilon),\tau]\right\}+\varepsilon^2\ldots,\end{aligned} \tag{17}$$

where

$$\begin{aligned}R_{ij}^{(\nu)}[\widetilde{W}_\sigma(\tau,\varepsilon),\tau]&=R_{ij}^{(\nu)}(\lambda_\sigma,\tau)E_{k_\sigma}+\varepsilon R_{i+1,j}^{(\nu)}(\lambda_\sigma,\tau)W_\sigma^{[0]}(\tau)\\&\quad+\varepsilon^2\left[R_{i+1,j}^{(\nu)}(\lambda_\sigma,\tau)W_\sigma^{[1]}(\tau)+\frac{1}{2}R_{i+2,j}^{(\nu)}(\lambda_\sigma,\tau)W_\sigma^{[0]2}(\tau)\right]+\varepsilon^3\ldots,\end{aligned}$$

$$R_{ij}^{(\nu)}(\lambda,t)\equiv\frac{\partial^{i+j}}{\partial\lambda^i\partial t^j}R^{(\nu)}(\lambda,t).$$

Substitute the series (11) and the series

$$\widetilde{\Lambda}_\sigma(\tau,\varepsilon)=\sum_{k=0}^{\infty}\varepsilon^k\Lambda_\sigma^{[k]}(\tau) \tag{18}$$

into (15) and equate the coefficients of ε^0. We obtain

$$(AK_\sigma^{[0]}\Lambda_\sigma^{[0]} - BK_\sigma^{[0]})G_\sigma^{[0]} = 0.$$

This equality becomes an identity when

$$K_\sigma^{[0]} \equiv K_\sigma, \qquad \Lambda_\sigma^{[0]} \equiv \Lambda_\sigma = \lambda_\sigma E_{k_\sigma}. \tag{19}$$

Taking (19) into account, equality (14) takes the form

$$\sum_{k=1}^{\infty} \varepsilon^{k-1}\Lambda_\sigma^{[k]} \cdot \sum_{k=0}^{\infty} \varepsilon^k G_\sigma^{[k]} = \sum_{k=0}^{\infty} \varepsilon^k G_\sigma^{[k]} \cdot \sum_{k=0}^{\infty} \varepsilon^k W_\sigma^{[k]} + \varepsilon \sum_{k=0}^{\infty} \varepsilon^k \frac{dG_\sigma^{[k]}}{d\tau}. \tag{20}$$

Equating in (15) the coefficients of ε, and in (20) those of ε^0, we obtain

$$\left[A\left(\frac{dK_\sigma}{d\tau} + K_\sigma\Lambda_\sigma^{[1]} + K_\sigma^{[1]}\lambda_\sigma\right) - BK_\sigma^{[1]}\right] G_\sigma^{[0]} = \sum_\nu a_\nu b_\nu K_\sigma R_{00}^{(\nu)}(\lambda_\sigma, \tau) G_\sigma^{[0]}, \tag{21}$$

$$\Lambda_\sigma^{[1]} G_\sigma^{[0]} = G_\sigma^{[0]} W_\sigma^{[0]}. \tag{22}$$

Equality (21), for any $G_\sigma^{[0]}$, becomes an identity if

$$\Lambda_\sigma^{[1]} = -M_\sigma D_\sigma^{[0]}, \qquad K_\sigma^{[1]} = P_\sigma D_\sigma^{[0]} + K_\sigma Q_{\sigma\sigma}^{[1]}, \tag{23}$$

where

$$D_\sigma^{[0]} = \frac{dK_\sigma}{d\tau} - A^{-1} \sum_\nu a_\nu b_\nu R_{00}^{(\nu)}(\lambda_\sigma, \tau) K_\sigma,$$

$$P_\sigma = \sum_{s \neq \sigma} K_s (\Lambda_s - \lambda_\sigma E_{k_s})^{-1} M_s,$$

$Q_{\sigma\sigma}^{[1]}$ is an arbitrary square matrix of order k_σ, differentiable the required number of times.

By assumption, $\Lambda_\sigma^{[1]}$ has no multiple eigenvalues. Let

$$G_\sigma = \left(G_1^{(\sigma)} \ldots G_{k_\sigma}^{(\sigma)}\right)$$

be the matrix transforming the matrix $\Lambda_\sigma^{[1]}$ to diagonal form

$$W_\sigma = \begin{bmatrix} w_1^{(\sigma)} & & 0 \\ & \ddots & \\ 0 & & w_{k_\sigma}^{(\sigma)} \end{bmatrix}.$$

Set

$$G_\sigma^{[0]} \equiv G_\sigma, \qquad W_\sigma^{[0]} \equiv W_\sigma. \tag{24}$$

Then (22) will be satisfied identically.

Suppose that $K_\sigma^{[0]}, \Lambda_\sigma^{[0]}, K_\sigma^{[1]}, \Lambda_\sigma^{[1]}, G_\sigma^{[0]}, W_\sigma^{[0]}, \ldots, K_\sigma^{[k-1]}, \Lambda_\sigma^{[k-1]}, G_\sigma^{[k-2]}, W_\sigma^{[k-2]}$ have already been determined.

Equating in equality (15) the coefficients of ε^k, and in equality (20) those of ε^{k-1}, we obtain

$$UK_\sigma^{[k]} = K_\sigma^{[k]}\Lambda_\sigma + K_\sigma\Lambda_\sigma^{[k]} + D_\sigma^{[k-1]}, \tag{25}$$

$$\Lambda_\sigma^{[1]}G_\sigma^{[k-1]} = G_\sigma^{[k-1]}W_\sigma + G_\sigma W_\sigma^{[k-1]} + T_\sigma^{[k-2]}. \tag{26}$$

The matrix $D_\sigma^{[k-1]}$ depends only on the quantities of the preceding approximations, while $T_\sigma^{[k-2]}$, in addition, depends on Λ_σ^{k-1}. The general solution of equation (25) has the form

$$\Lambda_\sigma^{[k]} = -M_\sigma D_\sigma^{[k-1]}, \qquad K_\sigma^{[k]} = P_\sigma D_\sigma^{[k-1]} + K_\sigma Q_{\sigma\sigma}^{[k]}, \tag{27}$$

where $Q_{\sigma\sigma}^{[k]}$ is an arbitrary square matrix of order k_σ. Let

$$G_\sigma^{[\mu]} = \left(G_1^{(\sigma)[\mu]} \ldots G_{k_\sigma}^{(\sigma)[\mu]}\right), \qquad W_\sigma^{[\mu]} = \begin{pmatrix} w_1^{(\sigma)[\mu]} & & 0 \\ & \ddots & \\ 0 & & w_{k_\sigma}^{(\sigma)[\mu]} \end{pmatrix},$$

$$T_\sigma^{[\mu]} = \left(T_1^{(\sigma)[\mu]} \ldots T_{k_\sigma}^{(\sigma)[\mu]}\right),$$

$$H_\sigma = \begin{bmatrix} H_1^{(\sigma)} \\ \vdots \\ H_{k_\sigma}^{(\sigma)} \end{bmatrix}$$

be the matrix inverse to the matrix G_σ. Then the general solution of the matrix equation (26) can be represented by the formulas

$$w_i^{(\sigma)[k-1]} = H_i^{(\sigma)} T_i^{(\sigma)[k-2]}, \qquad G_i^{(\sigma)[k-1]} = \Pi_i^{(\sigma)} T_i^{(\sigma)[k-2]} + G_i^{(\sigma)} q_{ii}^{(\sigma)[k-1]}$$

$$(i = 1, \dots k_\sigma), \tag{28}$$

where

$$\Pi_i^{(\sigma)} = \sum_{j \neq i} \frac{G_j^{(\sigma)} H_j^{(\sigma)}}{w_j^{(\sigma)} - w_i^{(\sigma)}},$$

and $q_{ii}^{(\sigma)[k-1]}$ is an arbitrary function. The arbitrariness in the choice of $Q_{\sigma\sigma}^{[k]}$ and $q_{ii}^{(\sigma)[k-1]}$ must, of course, be restricted by the condition of their differentiability.

The recurrence relations obtained make it possible successively to determine the terms of the series (11) and (18), by means of which the formal solution of equation (2) is represented. Retaining in these series a finite number of the first terms, we obtain an approximate solution of equation (2). Setting $\varepsilon = 1$ in the final formulas, we obtain an approximate solution of the original system (1).

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