Fredholm integral equations of the first kind

Authors: V. K. Ivanov

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Abstract

Full Text

Preamble

DIFFERENTIAL EQUATIONS

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On the Theory of Linear Integro-Differential Equations

In this paper, we investigate the properties of solutions for a class of linear integro-differential equations. The study of such equations is of significant importance in various fields of mathematical physics and engineering, particularly in problems involving hereditary mechanics and transport theory.

Consider the general form of the linear integro-differential equation:

$$\frac{dy}{dx} + A(x)y = \int_a^b K(x,s)y(s)ds + f(x)$$

where $A(x)$ and $f(x)$ are given functions, and $K(x,s)$ represents the kernel of the integral operator. Our objective is to establish existence and uniqueness theorems under specific boundary conditions, while also exploring the asymptotic behavior of the solutions as the domain extends.

1. Existence and Uniqueness of Solutions

To analyze the existence of solutions, we assume that the coefficients and the kernel satisfy certain regularity conditions. Specifically, let $A(x)$ be continuous on the interval $[a,b]$, and let the kernel $K(x,s)$ be square-integrable such that:

$$\int_a^b \int_a^b |K(x,s)|^2 dx ds < \infty$$

Under these assumptions, we can transform the integro-differential equation into an equivalent integral equation of the second kind. By applying the contraction mapping principle or the Fredholm alternative, we can demonstrate that for any continuous function $f(x)$, there exists a unique solution $y(x)$ that satisfies the initial condition $y(a) = y_0$.

2. Stability and Asymptotic Behavior

A critical aspect of the theory involves the stability of the solutions with respect to perturbations in the initial data and the non-homogeneous term $f(x)$. We utilize Lyapunov-like functionals to derive bounds on the solution. Suppose there exists a positive definite function $V(x, y)$ such that its derivative along the trajectories of the system satisfies:

$$\frac{dV}{dx} \leq -\alpha V(x,y) + \beta \int_a^x |y(s)| ds$$

where $\alpha > 0$ and $

INTRODUCTION

We consider the equation $\int_a^b K(x,t)u(t)dt = f(x)$ $(c < x < d)$, denoted as $Ku = f$. We assume the right-hand side of equation (1) always belongs to L_2. The kernel $K(x,t)$ is assumed to be square-summable and closed. We denote the functional space in which the solution $u(t)$ is sought as U. Let us define

$$B(t) = \left(\int_c^d |K(x,t)|^2 dx \right)^{1/2}. \quad (2)$$

We consider two cases in parallel:

1) The L_2 case: $B(t) \in L_2$, $U = L_2[a,b]$.
2) The C case: $B(t) \in C$, $U = C[a,b]$.

In case 2), we further assume that the adjoint operator K^*, corresponding to the transposed kernel $K^*(x,t) = K(t,x)$, maps every function from L_2 into a continuous function. We shall assume that for a given f, the equation has a solution (which is unique due to the closedness of the kernel) within some correctness class $\mathfrak{M}$ of the space U. However, instead of the exact function f, we are given an available function $f_\delta(x)$ such that

l l / o - / 6 | l < 6 , (3)

where $\delta > 0$ is a given parameter. Based on the given $f_\delta(x)$, we seek a function (an approximate solution to equation (1)) such that, within the metric of the space,

$u_\delta \to u_0$ as $\delta \to 0$. (4)

As is well known, due to instability, one cannot simply set $f(x) = f_\delta(x)$ in (1) to find the solution. Therefore, equation (1) is typically replaced by an equation of the second kind through the introduction of a small parameter α. We take as an approximation the solution $u(x; \alpha, f_\delta)$ of the equation $\alpha u + K^* K u = K^* f_\delta$, where $\alpha > 0$, given a relationship between α and δ that ensures (4). This is the simplest equation with a symmetric kernel, which can be derived by applying the methods described in [**?**, **?**, **?**, **?**].

Let us denote $\Delta(\alpha, \delta; u_0) = \sup \|u_0(x) - u(x; \alpha, f_\delta)\|$ for $\|f - f_\delta\| \leq \delta$. It is natural to pose the following problem: Let $\delta = \delta(\alpha)$ $(0 < \alpha < \alpha_0, \delta > 0)$ be a continuous function that increases monotonically with α. What must be the asymptotic behavior of this function as $\alpha \to 0$, and what must be the correctness class $\mathfrak{M}$ to which the exact solution u_0 belongs, such that

$\lim_{\alpha \to 0} \Delta(\alpha, \delta(\alpha); u_0) = 0$? (7)

In works [**?**, **?**, **?**, **?**], various systems of sufficient conditions for the correctness class $\mathfrak{M}$ and the function $\alpha(\delta)$ (the inverse of $\delta(\alpha)$) are provided to ensure (7). A. B. Bakushinsky [**?**] proved that the convergence of the ratio $\delta^2/\alpha \to 0$ is necessary and sufficient for the validity of (4) in the L_2 case. We present this result in a slightly different form in Theorem 10. In the present paper, the following results have been obtained in this direction. If we take as the approximate solution to equation (1) the function···

$u_\delta(t) = u(t, \alpha(\delta), \delta)$,

If $u(t, \alpha, \delta)$ is the solution to equation (5), then for the weak convergence in (4), it is necessary and sufficient that $\delta \to 0$ (Theorem 10). This assertion demonstrates that under weak regularization in the A. N. Tikhonov relation $\alpha(\delta) \leq C\delta$ [**?**], the exponent of 2 cannot be increased. In the C-case, let us denote by $\mathfrak{M}$ the closed linear span in the space of the complete orthonormal system of eigenfunctions.

$K_1(t, s) = \int_a^b K(x, t) K(x, s) dx, \quad a < t, s < b,$

corresponding to the operator A. It is always the case that $\mathfrak{M} \subseteq \mathcal{F}$. Theorem 5 provides a sufficient condition for the coincidence $\mathfrak{M} = \mathcal{F}$ under the assumption of continuity of $K_1(t, s)$. Theorem 11 establishes necessary and sufficient conditions for the function $\alpha(\delta)$ to ensure (7), provided that $u \in \mathfrak{M}$. It is shown that the class $\mathfrak{M}$ is determined by the summability of the Fourier series of the exact solution by a certain linear method (e.g., A-summability) with respect to the system $\{\phi_i\}$. The form of the function $\alpha(\delta)$ depends exclusively on the asymptotics of the operator norm and is not related to the properties of the exact solution.

§ 2. DECOMPOSITION OF THE DEVIATION

For $b = 0$, the function $u(t;\alpha, f)$ and $u(t;\alpha)$ serves as a solution to the equation $au + K^*Ku$. Let us define:

$$\Delta_1(\alpha; u_0) = \|u_0(t) - u(t;\alpha)\|,$$

$$\Delta_2(\alpha, \delta) = \sup_{\|f - f_\delta\| \leq \delta} \|u(t;\alpha) - u(t;\alpha, f_\delta)\|.$$

According to Ivanov, assuming $\|f\| \leq \delta$, we always define the norm for these functions within the metric of the space L_2. **Theorem.** For validity, it is necessary and sufficient that $\lim \Delta(\alpha; \alpha(\alpha, \delta(\alpha)) = 0$. **Proof.** The value $\Delta(\alpha, \delta; u_0)$ is non-decreasing with respect to δ; therefore:

$$\Delta_1(\alpha; u_0) = \Delta(\alpha, 0; u_0) \leq \Delta(\alpha, \delta; u_0), \quad \delta > 0.$$

The necessity of the first of conditions (10) follows directly from this. By applying the triangle inequality for norms to the identity

$$u_0(t) - u(t;\alpha, f_\delta) = [u_0(t) - u(t;\alpha)] + [u(t;\alpha) - u(t;\alpha, f_\delta)]$$

and taking the supremum over f such that $\|f - f_\delta\| \leq \delta$, we obtain:

$D(a, \delta; u_0) < D_x(a; \psi) + D_2(a > \delta),$

$A_2(a, b) < A(a, b; u_0) + D_x(a; u_0).$

The first of these inequalities implies the sufficiency of the relations in (10). Given that the first relation in (10) is satisfied, the second inequality establishes the necessity of the second relation in (10).

§ 3. ANALYTICAL REPRESENTATION OF THE SOLUTION

Solution Representation and Estimation

For the analytical representation of the solution, we employ the fundamental functions of E. Schmidt ([**?**, p. 136]). Let us denote the system of characteristic numbers for the positive symmetric kernels, defined by the operators K and K^*, as $\lambda_1 \leq \lambda_2 \leq ....$ The corresponding orthonormal systems of eigenfunctions are denoted by $\{\phi_k(x)\}$ and $\{\psi_k(t)\}$. These eigenfunctions are related by the following expressions:

$$\psi_k(t) = \lambda_k \int_a^b K(x,t)\phi_k(x)dx, \quad \phi_k(x) = \lambda_k \int_c^d K(x,t)\psi_k(t)dt \quad (k = 1, 2, \dots)$$

and they always belong to L_2. Furthermore, in the continuous case (C-case), the functions $\psi_k(t)$ are continuous, and the operator maps $[c, d]$ accordingly. By considering the relations in (11) as the Fourier coefficients of the function $K(x, t)$ for a fixed system $\{\psi_k(x)\}$ and applying Parseval' s identity, we can determine the properties defined in (2). In the C-case, the series on the left side of (12) converges uniformly according to Dini' s theorem. Let us now introduce the Fourier series:

$$u(x) = \sum_{k=1}^{\infty} c_k \phi_k(x)$$

(13)

$$u_0(t) = \sum_{k=1}^{\infty} a_k \psi_k(t)$$

(14)

Under the adopted conditions where $[c, d] \subseteq [a, b]$, these series converge in the mean with coefficients a_k and c_k. From this, we obtain the Picard expansion:

$$u_0(t) = \sum_{k=1}^{\infty} \lambda_k c_k \psi_k(t)$$

Solving equation (8) using the Schmidt method, we obtain:

$$u(x) = \sum_{k=1}^{\infty} \frac{a_k}{\lambda_k} \phi_k(x) = \sum_{k=1}^{\infty} \frac{c_k \lambda_k}{\lambda_k} \phi_k(x)$$

(16) From this result, we derive the following metric:

$\Delta_i(a; u_0) = \|u_0 - u(t; a)\| = \sqrt{\sum_{i=1}^{\infty} \frac{a^2 \lambda_i^2}{1 + a\lambda_i} X_i^2(t)}$

The limit relation $\lim \Delta = 0$ is equivalent to the metric of the space, specifically the relation

$\sum_{i=1}^{\infty} \frac{a^2 \lambda_i^2}{(1 + a\lambda_i)^2} X_i^2(t) \to u_0(t)$ as $a \to 0$

This can be regarded as the result of summing the Fourier series (14) using the convergence factors $1/(1 + a\lambda_i)$; we shall refer to this method as α-summation. In Theorems 2 and 4 below, we prove the regularity of this method. Since the series (13) converges in the mean, it follows that the series (16) for $u(t; a)$ also converges in the mean for $a > 0$.

Theorem. If the exact solution belongs to the space, then convergence in the mean holds:

$\lim_{a \to 0} u(t; a) = u_0(t)$.

Proof. Applying Parseval' s identity to (17), we obtain $\int |u_0(t) - u(t;a)|^2 dt$. Since the numerical series converges and the terms satisfy the condition for $0 < a < a_0$, the series on the right-hand side converges uniformly with respect to a. Consequently, the limit transition as $a \to 0$ is valid in (18).

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Corollary. The method of α-summation is regular with respect to convergence in the mean.

Definition.

1. Let us denote by

The class of functions in the space consists of those whose Fourier series with respect to system a are summable to these functions in the metric of the space. Given that if equation (1) has a solution in L, its Fourier series (14) converges to it in the mean, and applying the corollary of Theorem 2, we arrive at the following proposition.

Theorem: In the C-case, to which we now turn, the situation is more complex.

Lemma: In the C-case, the function $u(t, \alpha)$ (for $\alpha > 0$), which is a solution to equation (8), is continuous, and its Fourier series (16) converges to it uniformly.

Proof: Let the function $f \in [c, d]$ have a Fourier series of the form (13). The function $u = K^* f$ can be represented using the kernel corresponding to the operator K^*, which acts in $C[a, b]$. Then, according to the generalized Hilbert-Schmidt theorem (see [**?**], p. 318), its Fourier series

$$u_0(t) = \sum \frac{f_k}{\lambda_k} \phi_k(t)$$

converges to it uniformly. However, for $\alpha > 0$, the series (16) is equiconvergent with this series. The assertion of the lemma follows from this.

Theorem: In the C-case, if the Fourier series (14) of the solution converges uniformly, then the equality $u(t, \alpha) \to u(t)$ holds uniformly on $[a, b]$ as $\alpha \to 0$.

Proof: We consider $u(t, \alpha)$ and the terms of the series as functions of two variables defined in the rectangle $\Pi = \{a \le t \le b, 0 \le \alpha \le \alpha_0\}$. The terms of the series are continuous in Π, and under the conditions of the theorem, series (16) converges uniformly in Π with respect to the set of variables according to Abel' s test. It follows that the function $u(t, \alpha)$ is continuous in the aggregate of its variables and, by Cantor' s theorem, is uniformly continuous. Consequently, the limit (19) holds uniformly on $[a, b]$.

Corollary: The method of α-summability is regular with respect to uniform convergence.

6. D e f i n i t i o n

In the C-case, let $\mathfrak{M}$ denote the closed linear span of the system $\{\phi_k(t)\}$ under the C-metric. From the definition of the class $\mathfrak{M}$, it follows that $\mathfrak{M} \subset \mathcal{H}$. If the kernel $K(t,s)$ corresponding to the operator is continuous, one can provide sufficient conditions for the coincidence $\mathfrak{M} = \mathcal{H}$. Since the kernel $K(t,s)$ is positive, Mercer' s theorem applies in the case of continuity, and the following bilinear formula holds:

$$K(t,s) = \sum_{k=1}^{\infty} \lambda_k \phi_k(t) \phi_k(s)$$

where the series converges absolutely and uniformly. This representation allows for a more precise characterization of the functional space associated with the operator. Under these conditions, the structural properties of the kernel directly determine the completeness of the system $\{\phi_k(t)\}$ within the specified metric space.

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Furthermore, the series on the right-hand side converges uniformly with respect to the set of variables.

By analogy with [**?**], we introduce the functions $H(t,s;\alpha) = \sum M_l M_{\pm}$ and $L(t;\alpha) = \int |H(t,s;\alpha)| ds$, where $\alpha > 0$. Since the series in (21) and (22) are equiconvergent, the continuity of the terms ensures that the series (22) converges uniformly across the set of variables. Consequently, the function $L(t;\alpha)$ is continuous and serves as an analogue to the Lebesgue function for α-summation.

Theorem: In the C-case, if the kernel $K_x(t,s)$ is continuous, a sufficient condition for the coincidence of the limits is the existence of a constant A such that:

$L(t;\alpha) < A, \quad a < t < b.$

The proof follows the same methodology as the proof of the theorem on page 183 in [**?**].

Operator Norm Estimation

Let us denote···

R a = (aE + K*K)- l K*,

identity operator in the space. For $a > 0$, the operator is bounded and acts such that

l i m | | # a | | = + c°.

From equations (5) and (8), we have:

$$u(t, \alpha) - u(t, \alpha, f_\delta) = R_\alpha (f_0 - f_\delta),$$

from which it follows on the basis of

$$\Delta^2(\alpha, \delta) = \|R_\alpha \delta\|^2.$$

(24)

From this equality, it is evident that for the validity of $\Delta(\alpha, \delta(\alpha)) \to 0$, it is necessary and sufficient that as $\alpha \to 0$. To find the conditions for $\delta(\alpha)$ that ensure (25), it is essential to know the asymptotic behavior of $\|R_\alpha\|$ as $\alpha \to 0$. We shall denote the norm of the operator R_α in various cases as $\|R_\alpha\|$.

Theorem. In the case of

IVANOV, where $\alpha = -1, 2, ...$, the equality in this relation is achieved.

Proof. Let $h(x) = \sum h_i \phi_i$ be a function from $[c, d]$. Applying the operator R_α to h, we find its image. Using the definition of the operator norm and applying Parseval' s identity, we obtain:

$$\|R_\alpha h\|^2 = \sum \left(\frac{h_i}{1 + \alpha \lambda_i} \right)^2.$$

The function $\phi(\lambda) = \frac{1}{1+\alpha\lambda}$ reaches its maximum, and this maximum is equal to $\frac{1}{1+\alpha\lambda_1}$. From this, it follows that:

and from (30) it follows that (27). If we set $a = -i$ and

$f \in \Gamma$ at $t = \pi$, $h_t = I$

then (27A) is obtained. **Corollary.** For (25) to hold in the L-case, it is necessary and sufficient that

$\delta(a) = o(\sqrt{a})$ as $a \to 0$.

This proposition follows from the relation and the theorem. Suppose that for each $\epsilon > 0$, a function is constructed on $[c, d]$ such that

$|1/\alpha - f/a| < \delta$. (31)

Theorem. In the case of L^2, for the weak convergence

$[u(t; \alpha) - u(t; \alpha, f_\beta(\alpha))] \to 0$ weakly as $\alpha \to 0$

to hold for any dependence of the function on the parameter satisfying (31), it is necessary and sufficient that

$\beta(\alpha) = o(1/\alpha)$. (32)

Proof. Let us denote

$u(t;\alpha) = u_\alpha, f_\beta(\alpha) = u(t;\alpha, f_\beta(\alpha))$.

According to the Banach-Steinhaus theorem, for weak convergence to zero, it is necessary and sufficient that: 1) the norm $\|v(t;\alpha, f)\|$ is uniformly bounded; and 2) for every function $h(t)$ in a set $H \subset L_2[a,b]$ whose closed linear span coincides with $L_2[a,b]$, the relation $\int_a^b v(t;\alpha,f)h(t)dt \to 0$ holds. Let us first consider condition 2). We take the system of eigenfunctions of the operator as our set H. Due to the closure of the kernel $K_1(t,s)$ (see (20)) corresponding to this operator, the system $\{\phi_k(t)\}$ is complete, and its closed linear span coincides with $L_2[a,b]$. In this case, the integrals (33) are equal to the Fourier coefficients of the function $v(t;\alpha,f)$. If we denote the Fourier coefficients of the functions $f_\alpha(t)$ and $f_\delta(t)$ with respect to the system $\{\phi_k(t)\}$ as c_k^α and c_k^δ, then based on (16), we can write:

$$\int_a^b [v(t;\alpha) - v(t;\delta)]\phi_k(t)dt = \frac{\alpha c_k^\alpha - \delta c_k^\delta}{\lambda_k + \alpha} + ...$$

Using the Cauchy-Bunyakovsky-Schwarz inequality, we have:

$$|c_k^\alpha - c_k^\delta| = \left| \int_a^b [f_\alpha(x) - f_\delta(x)]\phi_k(x)dx \right| \leq$$

$$\leq \left(\int_a^b |f_\alpha(x) - f_\delta(x)|^2 dx \right)^{1/2} < \epsilon.$$

Thus, for any fixed t and any dependence $\delta = \delta(\alpha)$ such that $\lim_{\alpha\to 0}\delta(\alpha) = 0$, the left-hand side of (34) tends to zero. It follows that condition 2) is satisfied for any function $\delta(\alpha)$ meeting the requirements of Section 2. Therefore, for the theorem under consideration to hold, it is necessary and sufficient that condition 1) is satisfied. However, as follows from (9), (24), and Theorem 6, condition 1) is equivalent to (32). This establishes the validity of Theorem 7.

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Theorem

In the C-case, the following estimate for the operator norm holds: $\|\cdot\| = \max$.

Proof. Let $h(x) \in [c,d]$, $v = R$, and assume the expansions (28) and (29) for $o(f)$ hold. By applying the Cauchy-Bunyakovsky-Schwarz inequality to (29), we obtain:

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From this, we obtain equation (35).

Corollary. In the C-case, a sufficient condition for (25) to hold is that

$\sigma(a) = o(a)$ as $a \to 0$. (36)

As in Section 7, (36) follows from (24) and (35). While (35) provides an upper bound, it is possible to derive an exact expression for the norm of the operator R_a when it acts in the C-case.

Theorem. In the C-case, $\|R_a\|_C = ...$

Proof. Let us take the interval $[c, d]$ and set h such that $\|R_a\|_C = \|R_a\|$. If the Fourier series for the system $\{\phi_n(x)\}$ is given by (28), then the expansion of $\sigma(t, h)$ takes the form (29). We shall consider the value of $v(t, h)$ at a specific point as the value of a functional $v(t, h)$ depending on f and defined on $[c, d]$.

$\|F_t\| = \sup |v(t, h)|$ subject to $\|h\| \leq 1$.

$\|R^*\| = \sup \|v(t, \cdot)\| = \sup |v(t, h)|$

for $a < t \leq b$ and $\|h\| \leq 1$. (40)

Using (39) and (29), we can write

Kbit)

$r = 1$

That is, the value of the functional at each fixed value is determined by the scalar product of vectors with components belonging to the space L. It follows from this that

$\|F_t\|^2 = Y$

Let us introduce a function of u and the parameter λ, denoted as $1 + \text{cdf}$. Then, (41) can be represented as a Stieltjes integral of $(1 + e^\lambda)$. Integrating by parts and noting that

$\chi(u + v; t) = \sum \frac{\partial \chi}{\partial u} = \chi(u)$

From (2), we find $(1 + a\lambda^n)\frac{\partial}{\partial \lambda} g(\lambda; x) d\lambda$. By making the substitution $\lambda = r$, we obtain $(1 + r)$. Substituting this into (40) and using the notation from (38), we arrive at (37). As in Theorems 6 and 7, the following proposition holds.

Corollary. In the C-case, for the validity of the expression, it is necessary and sufficient that

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A. N. Tikhonov, *DAN SSSR*, 151, no. 5 (1963): 1023–1026.## § 5. Discussion

Convergence of the Approximate Solution to the Exact Solution

As demonstrated in Theorem 1, for the upper bound of the deviations to tend toward zero, it is both necessary and sufficient that the approximation conditions are satisfied. By combining these results, we can establish the formal convergence criteria for the numerical method.

In § 3 and 4, we will obtain conditions that ensure simi-

The convergence of the approximate solution $u(t, \alpha, f_\delta)$ to the exact solution $u_0(t)$ is addressed in the following theorems. In Theorems 10 and 11, it is assumed that the convergence $u(t, \alpha, f_\delta) \to u_0(t)$ as $\alpha \to 0$ must hold for any dependence of the function f_δ on the parameter δ such that $\|f_\delta - f\| < \delta$.

Convergence Theorems

Theorem 10. In the L_2-case, if equation (1) with $f(x) = f$ has a solution $u_0(t) \in L_2[a, b)$, then the following orders for the function $\delta(\alpha) \to 0$ are necessary and sufficient: 1. $\delta = o(\sqrt{\alpha})$ for strong convergence; 2. $\delta = O(\sqrt{\alpha})$ for weak convergence.

Proof. The proof follows from Theorems 1 and 2, in conjunction with either Theorem 6 or Theorem 7.

Theorem 11. In the C-case, if equation (1) with $f(x) = f$ has a solution $u_0(t) \in C[a, b]$, then for uniform convergence it is: 1. Sufficient that $\delta = o(\alpha)$; 2. Necessary and sufficient that $\delta = o\left(\frac{\alpha}{\omega(\alpha)}\right)$, where $\omega(\alpha)$ is defined by (38).

Proof. The proof follows from Theorem 1, Definition 1, and either Theorem 8 or Theorem 9.

The condition $\delta = o(\sqrt{\alpha})$ regarding strong convergence in Theorem 10, as well as the assertions in our Theorems 2 and 6, are contained in the work of A. B. Bakushinsky [?]. A special case of this theorem was also recently proved by Khudak [?].

From Sections 3 and 4, it follows that relations (42) and (43), which define the···

The quantities that determine the error in Theorem 1, as defined by (44), depend on entirely different factors. The value δ is associated with the ill-posedness of the problem; according to (24), its estimate is determined by the norm of the operator and requires no a priori information regarding the properties of the exact solution. In contrast, the value depends on the rate of convergence of the α-summation of the Fourier series of the function $u(t)$ to the function itself. For (42) to hold, it is necessary and sufficient that $u(t)$ belongs to the class $\mathfrak{M}$ (Definition 1), which in certain cases (Theorems 3 and 5) is defined quite simply. It is interesting to note that the validity of (44) does not require the compactness of the class $\mathfrak{M}$ at all; compactness does not even play an indirect role here, such as through compact embedding (see [?]). In the specific case of the Cauchy problem for the Laplace equation, this fact has been previously noted. If we do not wish to limit ourselves to the convergence in (44) but instead aim to find a quantitative estimate for $\Delta(\alpha, \delta)$, it is impossible to do so for the entire class $\mathfrak{M}$ due to its non-compactness, as the convergence in (42) is not uniform with respect to u on $\mathfrak{M}$. In this case, one must have in-

formation regarding the membership of $u(t)$ in some subclass Q of the class $\mathfrak{M}$, on which the convergence would be uniform with respect to $(\alpha; Q) = \sup \Delta$:

$$\lim_{\alpha \to 0} \Delta(\alpha; Q) = 0.$$

For this to hold, the compactness of the class Q is essential.

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On the Stability of Solutions to Differential Equations with Respect to Perturbations of the Right-Hand Side

A. N. Tikhonov *Proceedings of the Academy of Sciences of the USSR (DAN SSSR), 1023-1026*

Introduction

In the theory of differential equations, the question of how solutions behave under small changes to the equations themselves is of fundamental importance. We consider a system of differential equations where the right-hand side depends on a small parameter μ. Specifically, we investigate the behavior of the solutions as $\mu \to 0$, particularly in cases where the order of the system may change at the limit.

Problem Formulation

Consider the following system of differential equations:

$$\frac{dz}{dt} = f(z, y, t)$$
$$\mu \frac{dy}{dt} = F(z, y, t)$$

where z and y are vector functions of dimensions n and m, respectively. The parameter μ is a small positive constant. We are interested in the behavior of the solution $(z(t, \mu), y(t, \mu))$ as $\mu \to 0$.

As μ approaches zero, the second equation formally degenerates into an algebraic (or transcendental) equation:

$$0 = F(z, y, t)$$

This leads to the "degenerate" system:

$$\frac{dz}{dt} = f(z, y, t)$$
$$0 = F(z, y, t)$$

Stability and Convergence

The primary challenge lies in determining the conditions under which the solution of the original system converges to the solution of the degenerate system as $\mu \to 0$. This transition is not always guaranteed and depends heavily on the stability of the roots of the equation $F(z, y, t) = 0$.

Let $y = \phi(z, t)$ be a root of the equation $F(z, y, t) = 0$. We define this root as "stable" if, for a fixed z and t, the equilibrium point $y = \phi(z, t)$ of the "associated" system:

$$\frac{dy}{d\tau} = F(z, y, t)$$

9. I v a n o v

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Figures

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ON THE EXISTENCE OF A REACHABLE REGION

I. P. KARASEV

In this note, for a system of two nonlinear differential equations

$$dx/dt = f(x) + vu, \tag{1}$$

where

$$x = (\xi, \eta),\ dx/dt = (d\xi/dt,\ d\eta/dt),\ f(x) = (f_1(\xi, \eta), f_2(\xi, \eta))$$

— are variable vectors of the phase plane $\mathbf{R}^2$; $u = (u_1, u_2)$ is a constant vector of the plane $\mathbf{R}^2$; $v(t)$ is a piecewise-continuous scalar function, called an admissible control and satisfying the condition $|v(t)| \leqslant 1$, the following problem is posed: to investigate whether there exists an open region of initial data $D \subset \mathbf{R}^2$, from every point of which, moving according to the law (1), it is possible to reach the origin O of the plane $\mathbf{R}^2$ in a finite time. The region D will be called the *reachable region*, and the origin O — reachable.

The necessity of studying such a problem has already been noted in the literature, for example [1—5]. Works [6, 7] are also devoted to issues of controllability of nonlinear systems.

We assume that in some region H containing the point O, the conditions for the existence, uniqueness, and continuability of solutions to system (1) are satisfied for any admissible v, the function $f(x) = 0$ only when $x = 0$, and belongs to the class C_H^m.

The origin O is called *reachable in the small* for system (1) if, for an arbitrary neighborhood $S(O)$ of the point O, there is a neighborhood $V(O) \subset S(O)$ such that for any point $x_0 \in V(O)$, the trajectory $x(x_0, t_0, t, vu)$, $t > t_0$, reaches the origin in finite time under some admissible control $v(t)$, without leaving the neighborhood $S(O)$.

The origin O is called *unreachable in the small* for system (1) if there is a neighborhood $S(O)$ such that for an arbitrary neighborhood $V(O) \subset S(O)$, there is at least one point $x_0 \in V(O)$ such that the trajectory $x(x_0, t_0, t, vu)$, $t > t_0$, leaves $S(O)$ for any admissible v in finite time, without reaching the origin.

In what follows, along with system (1), we consider the equation:

$$\frac{d\eta}{d\xi} = \frac{f_2(\xi, \eta) + vu_2}{f_1(\xi, \eta) + vu_1}, \tag{2}$$

We will look for a solution η of equation (2) with constant $v \neq 0$ in a sufficiently small neighborhood of the origin, passing through the point O, in the form [8]

Figure 1: Figure 1

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$$\eta = k\xi + \sum_{s=2}^{m-1} \eta_0^{(s)} \frac{\xi^s}{s!} + r_m. \tag{3}$$

In the vicinity of point O, we have

$$\hat{f}_i(\xi, \eta) = \sum_{r=1}^{m-1} D' f_i(\xi, \eta) + R_m^{(2)}, \tag{4}$$

where $D^p f_i(\xi, \eta)$ — p-th order differential, divided by p! Substituting (3) and (4) into equation (2), we find the first coefficients of the expansion (3):

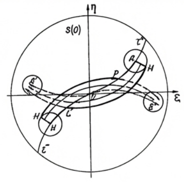

Fig. 1

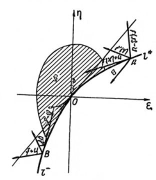

Fig. 2

$$k = \frac{u_2}{u_1}, \ \eta^{(2)} = \frac{1}{\nu u_1} d_2, \text{ and } d_2 = D^2 f_1(1, k) - kD^2 f_1(1, k).$$

If $d_2 = 0$, then $\eta^{(3)} = \dfrac{21}{\nu u_1} d_3$, where $d_3 = D^2 f_2(1, k) - kD^2 f_1(1, k)$. Let us also introduce notation: $d_1 = D^{S-1} f_3(1, k) - kD^{S-1} f_1(1, k)$, $(s = 2, 3, \ldots)$. It is easy to prove the following proposition: let it be proved for $d_i = 0$, $i < q$, that the solution η has the form

$$\eta = k\xi + \frac{d_q}{\nu u_1} \frac{\xi^e}{q} + \ldots$$

Then for $d_i = 0$, $i < q$, the equality also holds

$$\eta = k\xi + \frac{d_{q+1}}{\nu u_1} \frac{\xi^{q+1}}{q+1} + \ldots \tag{5}$$

Theorem 1. Let in equation (1) $f(x) \in C_H^m$ and $d_1 = d_2 = \ldots = d_{2s-1} = 0$, $d_{2s} \neq 0$, $2s \leqslant m$. Then the origin is reachable in the small for system (1).

Proof. From the conditions of the theorem it follows that the solution η, according to (5), has the form

$$\eta = k\xi + \frac{d_{2s}}{\nu u_1} \frac{\xi^{2s}}{2s} + \ldots$$

Figure 2: Figure 2

Then the trajectories of equation (1) in a sufficiently small neighborhood $S(O)$ are either convex upwards or convex downwards, and the trajectories of opposite families have opposite convexity (for $\upsilon = \pm \upsilon_0$).

Let us construct the curve l by definition

$$l = \begin{cases} l^+ \sim x(0, 0, t, u), & t \leqslant 0, \\ l^- \sim x(0, 0, t, -u), & t \leqslant 0. \end{cases} \tag{6}$$

The curve l divides $S(O)$ into two parts: S^+ and S^-. Let l^- be convex upwards, l^+ convex downwards. Consider their continuations for $t > 0$. Let us denote by S^- the part where the continuation of l^+ is located.

Let A, B be points on the semi-trajectories l^+, l^- and A', B' be point on their continuations (Fig. 1). By the theorem on the continuous dependence of solutions on initial data [9], for any neighborhood of solutions on an initial data [for any neighborhood $\gamma_{A'} \subset S^-$ of point A', one can construct a neighborhood γ_A of point A such that the trajectories $x(x_0, t_0, t, u)$, originating from the neighborhood γ_A, fall into the neighborhood $\gamma_{A'}$ in time T.

The reasoning for small neighborhoods of points B and B' is silmilar. Therefore, positive semi-trajectories originating from the part $S^+ \cap \gamma_A$ under control $\nu = 1$ will move from S^+ to S^-, crossing l^-. In exactly the same way, positive semi-semi-trajectories originating from the part $S^- \cap \gamma_B$ under control $\nu = -1$ will move from S^- to S^+, crossing l^+.

Let us draw, further, in the sets $S^+ \cap \gamma_A$ and $S^- \cap \gamma_B$ the diameters AM and BN.

Let $x(M, t_0, t, u)$, $t > t_0$, and $x(N, t_0, t, -u)$, $t > t_0$, cross l at points L and P. We have obtained a closed region $AMLBNPA$, which we denote by $V(O)$. It is not difficult to see that any point $x_0 \in V(O)$ rea–ches the origin O in a finite time without leaving $V(O)$, and therefore without leaving $S(O)$.

Theorem 1 is proved.

I at points L and P. We have obtained a closed region $AMLBNPA$, which we denote by $V(O)$. It is not difficult to see that any point x_0, $\in V(O)$ reaches the origin O in a finite time without leaving $V(O)$, and therefore without leaving $S(O)$.

Theorem 1 is proved.

Theorem 2. *Let* in equation (1) $f(x) \in C_H^m$ *and*

$$d_1 = d_2 = \ldots = d_{2s} = 0, \ d_{2s+1} \neq 0, \ 2s + 1 \leqslant m.$$

Then the origin is unreachable in the small for system (1).

Proof. The trajectory (5) has an inflection point at the origin. In this case, the curve (6) lies on one side of the straight line $\eta = k\xi$. Let it lie below the line the (Fig. 2). Take on l^+ an arbitrary point A and draw from point A the tangent vector $f(x) + u$ and the vectors u and $f(x)$. Then (Fig. 2) the vectors $f(x) + \nu u$, $\nu \neq 1$, are directed above the tangent vector. Similar reasoning can be carried out for m any point of the semi-trajectory l^-. It is not difficult to see that there exist points $x_0 \in S$ (Fig. 2) ɑtɒnst ʁdɘ ∝ 1 in other axɛr tsons that do not reach the origin in a finite time for $\nu =$ const. If $\nu \neq$ const, then at at each point all trajectories are still located between the vectors $f(x) + u$ and $f(x) - u$. Therefore, in this case too, it is impossible to reach the origin from points $x_0 \in S$.

It is easy to prove

Corollary. Let the functions $f_i(\xi, \eta)$ be homogeneous polynomials of degree r. If $d_{r+1} = 0$, then the origin O is unreachable in the small. If $d_{r+1} \neq 0$, then the origin O is reachable in the small for odd r and unreachable for even r.

In an obvious way it is proved

Theorem 3. *Let the origin be reachable in the small for system* (1). *Then the domain of attraction* Π *for* $\nu = 0$ (*in the sense of* Lyapunov *stability*) *is the domain of reachability*; in particular, if the solution $x = 0$ *is asymptotically stable globally, then the domain of reachability is the entire plane.*

Figure 3: Figure 3

Let us consider the following examples clarifying the introduced concepts.

Example 1. $d\xi/dt = -\xi + v$,
$d\eta/dt = -2\xi - 3\eta + 3\xi^2 - v$.

It is easy to prove that the origin is not reachable in the small, although the system is asymptotically stable in the large. The optimal control problem has no solution.

Example 2. $d\xi/dt = -\frac{1}{4}\xi + v$,
$d\eta/dt = 4a(1 + \frac{1}{3}\theta)\xi^3 - 16a\theta\xi^2 - \eta$,

where $a > 0$, $0 < \theta < 1$.
The curve l for the given system has the form

$$l = \begin{cases} l_+ \sim \eta = a(1 + \frac{2}{3}\theta)\xi^4 - \frac{16}{3}a\theta\xi^3, & t \leqslant 0, \\ l_- \sim \eta = -a\xi^4 + \frac{16}{3}a\theta\xi^3, & t \leqslant 0. \end{cases}$$

It is easy to show that the origin is not reachable in the small. But a reachable set exists: the trajectory l_- for $t > 0$ intersects the semi-trajectory l_+ outside the neighborhood $S(O)$.

Example 3. $d\xi/dt = \xi + v$,
$d\eta/dt = 3\xi + 4\eta + 4\xi^3 - v$.

The origin is reachable in the small for this system. We construct the boundary of the reachable set. The point

Example 3. $d\xi/dt = \xi + v$,
$d\eta/dt = 3\xi + 4\eta + 4\xi^3 - v$.
The origin is reachable in the small for this system. We construct the boundary of the reachable set. The point
$(-v; v + v; v^3)$ is the only unstable singular point of node type. For $v = 1$ and $v = -1$ we obtain, we obtain, respectively, singular points α_+ $(-1; 2)$ and $\alpha-$ $(1; -2)$. The switching curve has the form

$$l = \begin{cases} l_+ \sim \eta = \xi^4 - \xi, & t \leqslant 0, \\ l_- \sim \eta = -\xi^4 - \xi, & t \leqslant 0. \end{cases}$$

The semi-trajectories γ_- and γ_+ of our system, emeging respectively from points α_+ and $\alpha-$ under controls $v = -1$ and $v = 1$ and running for $t \leqslant 0$, have the form

$$\gamma_+ \sim \eta = \frac{1}{8}(7\xi^4 - 4\xi^3 - 6\xi^2 - 12\xi - 1),$$

$$\gamma_- \sim \eta = \frac{1}{8}(-7\xi^4 - 4\xi^3 + 6\xi^2 - 12\xi + 1).$$

The reachable set is the region bounded by the curve γ, consisting of semi-trajectories γ_+ and γ_-.

We see from Example 3 that for nonlinear systems, the reachable set can be non-convex.

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Figure 4: Figure 4

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Figure 5: Figure 5

moreover for $a = \frac{1}{\lambda_n^2}$ $(n = 1, 2, \dots)$ in this relation equality is achieved

$$\|R_a\|_{L_2} = \frac{1}{2\sqrt{a}}; \quad a = \frac{1}{\lambda_n^2}. \tag{27A}$$

Proof. Let

$$h(x) = \sum_{i=1}^{\infty} h_i \varphi_i(x) \tag{28}$$

be a function from $L_2[c, d]$. Applying the operator R_a to $h(x)$, we find

$$R_a h = \sum_{i=1}^{\infty} \frac{\lambda_i}{1 + a\lambda_i^2} h_i \psi_i(t). \tag{29}$$

Using the definition of the operator norm and applying Parseval's equality to (28) and (29), we obtain

$$\|R_a\|_{L_2}^2 = \sup \sum_{i=1}^{\infty} \left(\frac{\lambda_i}{1 + a\lambda_i^2}\right)^2 h_i^2 \text{ for } \sum_{i=1}^{\infty} h_i^2 \leqslant 1. \tag{30}$$

The function $g(\lambda) = \frac{\lambda}{1 + a\lambda^2}$ reaches a maximum for $\lambda = \frac{1}{\sqrt{a}}$ and this maximum is equal to $\frac{1}{2\sqrt{a}}$. Hence

$$\frac{\lambda_i}{1 + a\lambda_i^2} \leqslant \frac{1}{2\sqrt{a}},$$

and from (30) follows (27). If we take $a = \frac{1}{\lambda_n^2}$ and

$$h_i = \begin{cases} 1 & \text{for } i = n, \\ 0 & \text{for } i \neq n, \end{cases}$$

then from (30) we get (27A).

Corollary. *For (25) to hold in the L_2-case, it is necessary and sufficient that*

$$\delta(a) = o(\sqrt{a}) \text{ as } a \to 0.$$

This proposition follows from relation (26) and Theorem 6.

8. Assume that for each δ $(0 \leqslant \delta \leqslant \delta_0)$ a function function $f_\delta(x) \in L_2[c, d]$ is constructed such that

$$\|f_0 - f_\delta\| \leqslant \delta. \tag{31}$$

Theorem 7. *For weak L_2-convergence to take place in the L_2-case for any dependence of the function f_δ on the parameter δ satisfying* (31),

Figure 6: Figure 6

9. Theorem 8. In the C-case the following estimate holds: for the norm of the operator

$$\|R_\alpha\|_C \leqslant B/\alpha, \quad B = \max B(t), \tag{35}$$

$B(t)$ is defined by (2).

Proof. Let $h(x) \in L_2[c, d]$, $\|h\| \leqslant 1$, $v = R_\alpha h$. For $h(x)$ and $v(t)$ the expansions (28) and (29) hold. Applying the Bunyakovsky-Schwarz inequality to (29), (x_i, x), we obtain

$$\| \|_c h(x) = L_2 h = \sup_t \int_a^b h(x) \text{ and } v(t) \Big| \sup_t \left[\frac{4z^3}{\sqrt{\alpha}} \left\{ \sum_{\lambda_i > z/\sqrt{\alpha}} \frac{\psi_i^2(t)}{\lambda_i^2} \right] dz, \tag{36}$$

Hence (35) is obtained.

Corollary. In order for (25) to be fulfilled in the C-case, it is sufficient that $\delta(\alpha) = o(\alpha)$ as $\alpha \to 0$. (36)

As in point 7, (36) follows from (24) and (35). In (35) $\|R_\alpha\|_C$ is estimated from above.

One can give an exact expression for the norm of R_α, when it acts from L_2 into C.

Theorem 9. In the C-case $\|R_\alpha\|_C = \omega(\alpha)/\alpha$, (37)

$$\text{where } \omega^2(\alpha) = \sup_t \int_a^b \frac{4z^3}{(1 + z^2)^3} \left(\sum_{\lambda_i > z/\sqrt{\alpha}} \frac{\psi_i^2(t)}{\lambda_i^2} \right) dz. \tag{38}$$

Proof. Let us take $h(x) \in L_2[c, d]$ and set $v(t; h) = R_\alpha h$. Then $\|R_\alpha\|_C = \sup_h \|v(t; h)\|_C$, $\|h\| \leqslant 1$. If the Fourier series with respect to the system $\{\psi_i(x)\}$ for $h(x)$ is (28), then the expansion of $v(t; h)$ has the form (29). We shall consider the value $v(t; h)$ at point t as the value of a functional depending on t defined on $L_2[c, d]$;

$$v(t; h) = F_t h, \tag{39}$$

$\|F_t\| = \sup_h |v(t; h)|$ for $\|h\| \leqslant 1$,

$$\|R_\alpha\|_C = \sup_h \|v(t; h)\|_C = \sup_h \sup_t |v(t; h)| = \sup_t \sup_h |v(t; h)| = \sup_t \|F_t\| \quad \text{for } a \leqslant t \leqslant b, \|h\| \leqslant 1. \tag{40}$$

Using (39) and (29), we can write

Figure 7: Figure 7

$$F_t h = \sum_{i=1}^{\infty} \frac{\lambda_i \Psi_i(t)}{1+\alpha\lambda_i^2} h_i,$$

i.e., the value of the functional F_t for each fixed value of t is equal to the scalar product of vectors with components $\frac{\lambda_i \Psi_i(t)}{1+\alpha\lambda_i^2}$ and h_i, bunonging to the space ℓ_2. It follows from this, that

$$\|F_t\|^2 = \sum_{i=1}^{\infty} \left| \frac{\lambda_i \Psi_i(t)}{1+\alpha\lambda_i^2} \right|^2. \tag{41}$$

Let us introduce a function of λ and the parameter t

$$g(\lambda; t) = \sum_{\lambda_i = \lambda} \frac{\Psi_i^2(t)}{\lambda_i^2}.$$

Then (41) can be represented as a Stieltjes integral

$$\|F_t\|^2 = \int_0^{\infty} \frac{\lambda^4}{(1+\alpha\lambda_i^2)^2} dg(\lambda; t).$$

Integrating by parts and noting that

$$g(+\infty; t) = \sum_{i=1}^{\infty} \frac{\Psi_i^2(t)}{\lambda_i^2} = B^3(t)$$

(see (2)), we find

$$\|F_t\|^2 = \frac{B^2(t)}{\alpha^2} = \int_0^{\infty} \frac{4\lambda^3}{(1+\alpha\lambda_i^3)^2} g(\lambda; t)\, d\lambda.$$

Making the substitution $\sqrt{\alpha\lambda} = z$, we obtain

$$\|F_t\|^2 = \frac{1}{\alpha^2} \int_0^{\infty} \frac{4z^3}{(1+z^2)^3} \left[B^2(t) - g\left(\frac{z}{\sqrt{\alpha}}; t \right) \right] dz =$$

$$= \frac{1}{\alpha^3} \int_0^{\infty} \frac{4z^3}{(1+z^2)^3} \left(\sum_{\lambda_i + \frac{\alpha}{z_i^3}} \frac{\Psi_i^2(t)}{z_i^2} \right) dz.$$

Substituting this into (40) and using the notation (38), we arrive at (37). Just as in Theorems 6 and 7, the following proposition holds.

Corollary. In the C-case, for (25) to be valid, it is necessary and sufficient, that

$$\delta(\alpha) = o\left(\frac{\alpha}{\omega(\alpha)} \right).$$

Figure 8: Figure 8

information about the belonging of $u_0(t)$ to a certain subclass Q of class M, on which the convergence (42) would be uniform with respect to $u_0(t)$, so that

$$\overline{\Delta}_1(\alpha; Q) = \sup \Delta_1(\alpha; u_0) \text{ for } u_0 \in Q$$

and

$$\lim_{\alpha \to 0} \overline{\Delta}_1(\alpha; Q) = 0.$$

For this, the compactness of the class Q is essential.

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named after V. A. Steklov

Figure 9: Figure 9