ON THE SCHWARZ FORMULA FOR THE EQUATION $\partial w/\partial \bar z+b\bar w=0$

MATHEMATICS

1967

SovietRxiv

Abstract

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UDC 517.9

MATHEMATICS

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ON THE SCHWARZ FORMULA FOR THE EQUATION $\partial w/\partial \bar{z} + b\bar{w} = 0$

(Presented by Academician I. N. Vekua on 12 I 1967)

1. Consider the differential equation

$$\partial f/\partial \bar{z} + Af + B\bar{f} = F, \tag{1}$$

where $\partial/\partial \bar{z} = \frac{1}{2}(\partial/\partial x + i\partial/\partial y)$, $f(z) = u(x,y) + iv(x,y)$, equivalent to a system of two elliptic-type equations with respect to the real-valued functions $u(x,y)$ and $v(x,y)$. The theory of equations (1) was created in the works of I. N. Vekua ([1,2]). In the case where $A, B, F \in L_p$, $p > 2$, the method of the book ([1]) is applicable, based on integral representations of solutions in terms of analytic functions of the variable $z = x + iy$. In the case where A, B, F are analytic in x and y, the method of the article ([2]) is applicable, based on the continuation of solutions into the domain of complex variables $z = x + iy$ and $\zeta = x - iy$. Sometimes, as, for example, in the case of constant coefficients A and B in an unbounded domain, the second method has advantages.

Introducing a new function by the formula

$$f(z) = w(z) \exp\left\{-\int A(z,\bar{z})\, d\bar{z}\right\}$$

and taking $F = 0$, from (1) we obtain

$$\partial w/\partial \bar{z} + b\bar{w} = 0, \tag{2}$$

where

$$b = B \exp\left\{2i \operatorname{Im} \int A(z,\bar{z})\, d\bar{z}\right\}.$$

Below the case of constant b is considered. As follows from ([1]), p. 182, every solution of equation (2) satisfies the metaharmonic equation

$$\partial^2 w/\partial z\,\partial \bar{z} - |b|^2 w = 0, \tag{3}$$

and conversely, if $w_1(z)$ is the general solution of equation (3), then

$$w(z) = w_1(z) - \frac{1}{b}\,\frac{\partial \bar{w}_1(z)}{\partial z}$$

gives the general solution of equation (2).

2. In ([1]), p. 308, a generalized Schwarz formula was obtained for the homogeneous equation of the form (1). The purpose of the present note is, using the generalized Cauchy formula obtained in ([3]) for the case of equation (2) with constant coefficient $b \neq 0$, to express in terms of special functions the kernels of the generalized Schwarz formula for canonical domains–the half-plane and the strip. Unfortunately, in the case of the disk it has not yet been possible to obtain an analogous formula, and the kernel is expressed in the form of a series (see formula (20)).

In the case of the half-plane the problem is to find a solution $w(z)$ of equation (2), regular in the upper half-plane $y \geq 0$, satisfying on the real axis the condition $\operatorname{Re} w = u(x)$, where $u(x)$ is a prescribed function, and satisfying the relation

$$w(z) = \exp\left[2|b|r\right] r^{-1/2} o(1), \qquad r = |z|, \tag{4}$$

in a neighborhood of the infinitely distant point. We shall assume here that $u(x)$ is once H-continuously differentiable and belongs to $L_2(-\infty, \infty)$.

Following ([1]), p. 239, we shall seek the solution $w(z)$ in the form of an integral of Cauchy type.

$$w(z) = \frac{1}{2\pi i}\int_{-\infty}^{\infty} \Omega_1(z,t) w(t)\,dt - \Omega_2(z,t)\overline{w(t)}\,dt, \qquad w(t) = u(t) + iv(t), \tag{5}$$

where (see ([3]))

$$\Omega_1(z,t) = \pi|b|\frac{\bar{z}-\bar{t}}{|z-t|} H_1^{(1)}(2i|b|\,|z-t|),$$

$$\Omega_2(z,t) = -\pi b i H_0^{(1)}(2i|b|\,|z-t|). \tag{6}$$

If in formula (5) we substitute the values $\Omega_1(z,t)$ and $\Omega_2(z,t)$, pass to the limit as $z \to x + i0$, and then use the boundary condition, then for the unknown function $v(t)$ we obtain the integral equation

$$\int_{-\infty}^{\infty} \left\{ |b| \frac{x-t}{|x-t|} H_1^{(1)}(2i|b|\,|x-t|) - b\cos\sigma \cdot iH_0^{(1)}(2i|b|\,|x-t|) \right\} v(t)\,dt =$$

$$= u(x) - \int_{-\infty}^{\infty} |b| \sin\sigma \cdot iH_0^{(1)}(2i|b|\,|x-t|) u(t)\,dt, \qquad (7)$$

where $b = |b|e^{i\sigma}$.

In doing this we have taken into account that $H_1^{(1)}(2i|b|\,|x-t|)$ is real, while $H_0^{(1)}(2i|b|\,|x-t|)$ is a purely imaginary quantity (see ([4]), p. 163). We may assume (see ([1]), p. 228) that the sought function $v(x)$ is H-continuously differentiable at every finite point of the real axis.

Performing obvious transformations on the left-hand side of equation (7), we obtain

$$\int_{-\infty}^{\infty} H_0^{(1)}(2i|b|\,|x-t|)\varphi(t)\,dt = u(x), \qquad -\infty < x < \infty, \qquad (8)$$

where

$$\varphi(t) = -\frac{1}{2i} v'(t) - |b|\cos\sigma \cdot iv(t) + |b|\sin\sigma \cdot iu(t). \qquad (9)$$

If $\varphi(t)$ has been found, then, solving the first-order linear differential equation (9), we obtain:

$$v(t) = \exp[2|b|\cos\sigma \cdot t] \left\{ C + \int [-2|b|\sin\sigma \cdot u(t) - 2i\varphi t] \times \right.$$

$$\left. \times \exp[2|b|\cos\sigma \cdot t]\,dt \right\}, \qquad (10)$$

where C is a real constant.

Equation (8) is solved by the method of Fourier integral transforms (see ([5]), pp. 400-401). It is easy to show that for $b \neq 0$ the kernel of equation (8) belongs to $L_1(-\infty, \infty)$ (see ([6]), p. 746, formula 14). Direct computations show that (for real b)

$$\varphi(x) = \frac{i}{4\pi} \int_{-\infty}^{\infty} \sqrt{\alpha^2 + 4b^2} \exp[-i\alpha x]\,d\alpha \int_{-\infty}^{\infty} u(t) \exp[i\alpha t]\,dt.$$

Substituting this expression into formula (10), we obtain:

$$v(x) = C\exp[2bx] +$$

$$+\frac{1}{i}\int_{-\infty}^{\infty}\left[\frac{\partial}{\partial x}H_0^{(1)}(2i|b|\,|x-t|) + bH_0^{(1)}(2i|b|\,|x-t|)\right]u(t)\,dt. \qquad (11)$$

Substituting (11) with $C = 0$ into formula (5), we obtain

$$w(z) = \frac{1}{2}\int_{-\infty}^{\infty}\left\{\frac{\partial}{\partial z}H_0^{(1)}(2i|b|\,|z-t|) + bH_0^{(1)}(2i|b|\,|z-t|)\right\}u(t)\,dt +$$

$$+\frac{1}{2}\int_{-\infty}^{\infty}u(t)\,dt\int_{-\infty}^{\infty}\left\{\frac{\partial}{\partial \bar{z}}H_0^{(1)}(2i|b|\,|z-x|) - bH_0^{(1)}(2i|b|\,|z-x|)\right\}\times$$

$$\times\left\{\frac{\partial}{\partial x}H_0^{(1)}(2i|b|\,|x-t|) + bH_0^{(1)}(2i|b|\,|x-t|)\right\}dx. \qquad (12)$$

Let us now consider the function

$$w(z) = \int_{-\infty}^{\infty}\left\{\frac{\partial}{\partial z}H_0^{(1)}(2i|b|\,|z-t|) + bH_0^{(1)}(2i|b|\,|z-t|)\right\}u(t)\,dt. \qquad (13)$$

If $u(t)$ vanishes in a neighborhood of the point at infinity, then (13) has the asymptotics (4). In addition, a direct verification shows that this function satisfies equation (2) (see the end of § 1) and the boundary condition $\operatorname{Re} w = u(x)$ on the axis $y = 0$. On the other hand, it can be represented by formula (12). Consequently, formula (12) coincides with formula (13) if $u(t)$ vanishes in a neighborhood of ∞. For arbitrary $u(t) \in L_2(-\infty, \infty)$, formula (13) is proved by passage to the limit.

Formula (13) is the **generalized Schwarz formula** for equation (2) in the case of a real constant b for the upper half-plane.

In the case of complex b, the formula has a more cumbersome form.

3. We consider the case of the strip $0 \le \operatorname{Im} z \le h$.

Seeking a solution $w(z)$ of equation (2), satisfying the conditions

$$\operatorname{Re} w = u_0(x) \quad \text{for } y = 0, \qquad u_0(x) \in L_2(-\infty, \infty),$$

$$\operatorname{Re} w = u_1(x) \quad \text{for } y = h, \qquad u_1(x) \in L_2(-\infty, \infty),$$

in the form of a Cauchy-type integral and arguing analogously to the preceding case, we obtain the following system of integral equations:

$$\int_{-\infty}^{\infty} H_0^{(1)}(2i|b|\,|x-t|)\varphi(t)\,dt + \int_{-\infty}^{\infty} H_0^{(1)}(2i|b|\,|x-t-ih|)\psi(t)\,dt = f_1(x),$$

$$\int_{-\infty}^{\infty} H_0^{(1)}(2i|b|\,|x-t|)\psi(t)\,dt + \int_{-\infty}^{\infty} H_0^{(1)}(2i|b|\,|x-t+ih|)\varphi(t)\,dt = f_2(x), \quad (14)$$

where

$$\varphi(t) = -\frac{1}{2i}v_0'(t) - biv_0(t), \qquad \psi(t) = -\frac{1}{2i}v_1'(t) - biv_1(t),$$

$$f_1(x) = u_0(x) - |b|h\int_{-\infty}^{\infty} H_1^{(1)}(2i|b|\,|x-t-ih|)\frac{u_1(t)\,dt}{|x-t-ih|},$$

$$f_2(x) = u_1(x) + |b|h\int_{-\infty}^{\infty} H_1^{(1)}(2i|b|\,|x-t+ih|)\frac{u_0(t)\,dt}{|x-t+ih|};$$

$w_0 = u_0 + iv_0$, $w_1 = u_1 + iv_1$ are the values of the sought function for $y = 0$ and $y = h$, respectively.

Solving system (14) by the Fourier method, we obtain the formula

$$\begin{aligned} w(z) = \frac{1}{2}\int_{-\infty}^{\infty}\Bigg\{ &\frac{\partial}{\partial z}H_0^{(1)}(2i|b|\,|z-t|) + bH_0^{(1)}(2i|b|\,|z-t|) \\ &+ \frac{1}{2\pi i}\int_{-\infty}^{\infty}\left[\frac{\partial}{\partial z}H_0^{(1)}(2i|b|\,|z-t_1|) - bH_0^{(1)}(2i|b|\,|z-t_1|)\right]R_1(t_1,t)\,dt_1 \\ &- \frac{1}{2\pi i}\int_{-\infty}^{\infty}\left[\frac{\partial}{\partial z}H_0^{(1)}(2i|b|\,|z-t_1-ih|)\right. \\ &\qquad \left. -bH_0^{(1)}(2i|b|\,|z-t_1-ih|)\right]R_4(t_1,t)\,dt_1\Bigg\}u_0(t)\,dt \\ +\frac{1}{2}\int_{-\infty}^{\infty}\Bigg\{ &\frac{\partial}{\partial z}H_0^{(1)}(2i|b|\,|z-t-ih|) + bH_0^{(1)}(2i|b|\,|z-t-ih|) \\ &- \frac{1}{2\pi i}\int_{-\infty}^{\infty}\left[\frac{\partial}{\partial z}H_0^{(1)}(2i|b|\,|z-t_1|) - bH_0^{(1)}(2i|b|\,|z-t_1|)\right]R_2(t_1,t)\,dt_1 \\ &+ \frac{1}{2\pi i}\int_{-\infty}^{\infty}\left[\frac{\partial}{\partial z}H_0^{(1)}(2i|b|\,|z-t_1-ih|) - bH_0^{(1)}(2i|b|\,|z-t_1+ih|)\right] \\ &\qquad \times R_3(t_1,t)\,dt_1\Bigg\}u_1(t)\,dt, \end{aligned} \quad (15)$$

where

$$\left.\begin{matrix} R_1(x,t) \\ R_3(x,t) \end{matrix}\right\} = \int_{-\infty}^{\infty} \left\{ 1 \pm \sqrt{\frac{2h}{\pi}}\,(\alpha^2+4b^2)^{1/4} \exp[-h\sqrt{\alpha^2+4b^2}] \right. \\ \left. \times K_{1/2}\Big(h\sqrt{\alpha^2+4b^2}\Big) \right\} \frac{(2b-i\alpha)\exp[i\alpha(t-x)]\,d\alpha}{(1-\exp[-2h\sqrt{\alpha^2+4b^2}])\sqrt{\alpha^2+4b^2}},$$

$$\left.\begin{matrix} R_2(x,t) \\ R_4(x,t) \end{matrix}\right\} = \int_{-\infty}^{\infty} \left\{ 1 \mp \sqrt{\frac{2h}{\pi}}\,(\alpha^2+4b^2)^{1/4} \exp[h\sqrt{\alpha^2+4b^2}] \right. \\ \left. \times K_{1/2}\Big(h\sqrt{\alpha^2+4b^2}\Big) \right\} \frac{(2b-i\alpha)\exp[-h\sqrt{\alpha^2+4b^2}]\exp[i\alpha(t-x)]\,d\alpha}{(1-\exp[-2h\sqrt{\alpha^2+4b^2}])\sqrt{\alpha^2+4b^2}}.$$

The upper function corresponds to the upper sign, the lower one to the lower sign. The case of the unit disk is considered analogously. However, the solution is not obtained in closed (integral) form, but by means of certain infinite series.

In conclusion I express my gratitude to I. I. Danilyuk for posing the problem and for assistance in writing the paper.

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Received
12 I 1967

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Note: Figure translations are in progress. See original paper for figures.