A GENERALIZED RIEMANN EQUATION IN THE THEORY OF DIRICHLET SERIES

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Abstract

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A GENERALIZED RIEMANN EQUATION IN THE THEORY OF DIRICHLET SERIES

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In the last decade, especially after the investigations of Hardy and Littlewood, many papers have appeared devoted to approximate and shortened equations for L-functions and to their applications in analytic number theory, for example ([1]).

In this article the author proves that shortened equations exist for a very broad class of functions expandable in Dirichlet series and satisfying a system of functional equations of Riemann type. The author' s results do not overlap with investigations known up to now in this area ([2]).

1. Let the functions $\varphi_i(s)$, $\psi_i(s)$ $(i = 1, 2)$ be meromorphic in the whole plane. The singularities of the functions $\varphi_i(s)$ may lie only in the half-strip $|t| \leq t_0$, $\sigma < a_1$, and the singularities of $\psi_i(s)$ in the half-strip $|t| \leq t_1$, $\sigma > a_2$. Moreover, for any pair of real numbers $\alpha < \beta$, the functions $\varphi_i(s)$, $\psi_i(s)$ tend uniformly to zero on the segments $\sigma \in [\alpha, \beta]$, $|t| \to \infty$. The functions $\varphi_i(s)$ and $\psi_i(s)$ are absolutely integrable along every vertical line not passing through singularities of these functions.

2. Let the functions $L_i(s)$, $i = 1, 2$, also be meromorphic in the whole plane and expandable in Dirichlet series:

$$L_i(s) = \sum_{n=1}^{\infty} \frac{a_{in}}{\nu_{in}^s}, \tag{1}$$

a_i are the abscissae of absolute convergence of the series (1).

In the whole plane, except for isolated singularities, the equations

$$\varphi_1(s)L_1(s) = \psi_1(s)L_2(k-s), \quad \varphi_2(s)L_2(s) = \psi_2(s)L_1(k-s), \quad k > 0 \tag{2}$$

hold.

In what follows we shall consider only the case when $k - a_2 < a_1$, since in the contrary case the theorem proved below is trivial. We shall call the strip $k - a_2 < \sigma < a_1$ critical.

3. The functions $\xi_i(s) = \varphi_i(s)L_i(s)$, $i = 1, 2$, are regular outside the critical strip.

If the functions $\xi_i(s)$ have infinitely many poles, then we require that there exist an infinite sequence of numbers T_n, $n = \pm1, \pm2, ...$, such that $|T_n| \to \infty$ as $n \to \infty$, and on the segments $\{\sigma + iT_n,\ a_1 \le \sigma \le a_2\}$, $a_1 < k - a_2$, $a_2 > a_1$, the condition $\max|\xi_i(s)| \to 0$ be fulfilled.

Theorem. *The formulas hold*

$$\xi_1(s) = \sum_{n=1}^{\infty} \frac{a_{1n}}{2\pi i} \int_{(\alpha)} \frac{\varphi_1(w)\nu_{1n}^{-w}}{w - s}\, dw + \sum_{n=1}^{\infty} \frac{a_{2n}}{2\pi i} \int_{(\alpha)} \frac{\psi_1(k - w)\nu_{2n}^{-w}}{w - (k - s)}\, dw + \lim_{n\to\infty} \sum_{\mu=-n}^{+n} Q_{1\mu}(s), \tag{3}$$

$$\xi_2(s) = \sum_{n=1}^{\infty} \frac{a_{2n}}{2\pi i} \int_{(\alpha)} \frac{\varphi_2(w)\nu_{2n}^{-w}}{w - s}\, dw + \sum_{n=1}^{\infty} \frac{a_{1n}}{2\pi i} \int_{(\alpha)} \frac{\psi_1(k - w)\nu_{1n}^{-w}}{w - (k - s)}\, dw + \lim_{n\to\infty} \sum_{\mu=-n}^{n} Q_{2\mu}(s),$$

where

$$Q_{kn} = \sum_{\tau_{n-1} \le \tau_m \le \tau_n} g_k\left(\frac{1}{s - \rho_m}\right), \qquad \rho_m = \sigma_m + i\tau_m, \qquad k = 1, 2;$$

$g_k(x)$ are the principal parts of the Laurent expansions, near the pole ρ_m, of the functions $\xi_k(s)$.

In the case of entire functions the third term in (3) is equal to zero. We shall confine ourselves to proving formula (3) for $\xi_1(s)$. Taking a sufficiently large, by Cauchy' s theorem we obtain

$$\xi_1(s) = \frac{1}{2\pi i} \int_{(a)} \frac{\varphi_1(w)L_1(w)}{w - s}\, dw - \frac{1}{2\pi i} \int_{(k-a)} \frac{\varphi_1(w)L_1(w)}{w - (k - s)}\, dw + \lim_{n\to\infty} \sum_{\mu=-n}^{n} Q_{1\mu}(s). \tag{4}$$

We use the functional equation (2) and make the corresponding change of variables; then

$$\xi_1(s) = \frac{1}{2\pi i} \int_{(a)} \frac{\varphi_1(w)L_1(w)}{w - s}\, dw + \frac{1}{2\pi i} \int_{(a)} \frac{\psi_1(k - w)L_2(w)}{w - (k - s)}\, dw + \lim_{n\to\infty} \sum_{\mu=-n}^{n} Q_{1\mu}(s). \tag{5}$$

As a increases, the first and second terms remain invariant. This circumstance allows us, using formula (1), to interchange the sign of summation and integration and to establish that the first two terms represent entire functions. Thus formula (3) for $\xi_1(s)$ is obtained.

The first two series of formula (3) can be combined into a single everywhere convergent series with an estimate of the remainder $O(\nu_{in}^{-A})$, where A is any positive number.

Formulas (3) can be extended to quite general classes of functions. Thus, for example, they can be applied to the functions considered by Hecke ([3]). In this case one obtains formulas generalizing those known to us ([4]) with a good remainder term. Formulas (3) are also applicable to L-functions of fields of algebraic numbers and of function fields, to ratios of these functions, etc. We give an example of formulas (3) for the ratio of two L-functions of the same parity:

$$\frac{L(s,\chi)}{L_1(s,\chi_1)} = \sum_{n=1}^{\infty} \frac{c_n}{n^s} \left\{ s_1 + \frac{\sqrt{s_1^2}}{\sqrt{\pi}} \gamma\left(\frac{1}{2}, s_1^2\lambda\right) \right\} +$$

$$+\Delta^{1/2-s} \sum_{n=1}^{\infty} \frac{c_n}{n^{1-s}} \left\{ s_2 + \frac{\sqrt{s_2^2}}{\sqrt{\pi}} \gamma\left(\frac{1}{2}, s_2^2\lambda\right) \right\} + e^{-\lambda(s-1/2)^2} \lim_{n\to\infty} \sum_{\nu=-n}^{n} Q_\nu(s), \quad (6)$$

$$\gamma(\alpha,s) = \int_0^s e^{-x} x^{\alpha-1}\,dx, \qquad s_1 = \frac{\ln u}{2\lambda} + s - \frac{k}{2}, \qquad s_2 = \frac{\ln u}{2\lambda} - s + \frac{k}{2},$$

$$u = \frac{A}{n}, \qquad A > 0,$$

Δ is the ratio of the moduli of the characters χ and χ_1. The convergence of the last term in (6) is connected with the problem of the density of the distribution of the zeros of the denominator, which even for the zeta-function has not yet been solved.

If one considers the ratio of two L-functions of a function field k with field of constants q, then the same equation is obtained, except that in place of the factor $\Delta^{1/2-s}$ there will be a constant quantity. In this case the series composed of the principal parts of the function will converge absolutely.

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Note: Figure translations are in progress. See original paper for figures.