ON EXTREMAL QUASICONFORMAL MAPPINGS WITH A PRESCRIBED BOUNDARY CORRESPONDENCE

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Abstract

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MATHEMATICS

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ON EXTREMAL QUASICONFORMAL MAPPINGS WITH A PRESCRIBED BOUNDARY CORRESPONDENCE

(Presented by Academician M. A. Lavrent' ev, 5 X 1966)

1. A quasiconformal homeomorphism $w = f(z)$ of the disk $U: \ |z| < 1$ onto the disk $U': \ |w| < 1$ with (measurable) characteristic $\mu(z) = w_{\bar{z}}/w_z$ (related to the characteristics $p(z), \theta(z)$ of M. A. Lavrent' ev ([1]) by the relation $\mu = -[(p-1)/(p+1)]e^{2i\theta}$), for which

$$\operatorname*{vrai\,max}_{z \in U} |\mu(z)| = k < 1,$$

i.e., the maximal dilatation $K[f] = (1+k)/(1-k) < \infty$, induces a homeomorphism $\tau = \omega(t)$ of the circle $S: \ |t| = 1$ onto $S': \ |\tau| = 1$, satisfying the known conditions (see ([2])). Let Ω be the set of all homeomorphisms $\tau = \omega(t): S \to S'$ admitting a quasiconformal extension into the disk U, and let $Q(\omega)$ be the class of all quasiconformal mappings $w = f(z): U \to U'$ with the restriction $f|_S = \omega$, for a given $\omega \in \Omega$. In this note we consider properties of mappings for which

$$K[f_0] = \inf_{f \in Q(\omega)} K[f].$$

It is known ([3–6]) that in the class of quasiconformal homeomorphisms $w = f(z)$ of the disk U onto U', carrying prescribed boundary points $z_1, \dots, z_n, \ |z_i| = 1$, to prescribed points $w_1, \dots, w_n, \ w_i = f(z_i)$, there exists, and moreover uniquely, a mapping with least maximal dilatation $K[f]$. This extremal mapping has a characteristic of the form $\mu(z) = k\varphi(z)/|\varphi(z)|$, where $k = \text{const}$, and $\varphi(z)$ is a function analytic in the disk $|z| \le 1$, except, possibly, for the points $z_1, \dots, z_n$, where it may have simple poles*. If, however, one considers quasiconformal mappings with a prescribed boundary correspondence ω, then in this class there may exist extremal mappings which no longer possess analogous properties. Indeed, as an example constructed in ([7]), no. 7, shows, there exist homeomorphisms $\omega \in \Omega$ such that in the class $Q(\omega)$ there are, moreover distinct, extremal mappings for which $|\mu(z)| \ne \text{const}$, as well as such mappings that $|\mu(z)| = k = \text{const}$, but $\mu(z)$ cannot be represented in the form $\mu(z) = k\overline{\varphi(z)}/|\varphi(z)|, \ \varphi \in A(U)$, where $A(U)$ denotes the Banach space of analytic functions in the disk U with finite norm

$$\|\varphi\|_{A(U)} = \iint_U |\varphi(z)|\,dx\,dy, \quad z = x + iy.$$

A quasiconformal mapping $w = f_0(z) : U \to U'$ with characteristic of the form

$$\mu(z) = k\overline{\varphi(z)}/|\varphi(z)|, \quad k = \text{const}, \quad \varphi \in A(U), \tag{1}$$

will be called a **Teichmüller mapping**. In ([8]) it is proved that the Teichmüller mapping $f_0(z)$ is the unique extremal mapping in the class $Q(\omega)$, where $\omega = f_0|_S$.

2. For a given $\omega \in \Omega$, put

$$K_0[\omega] = \inf_{f \in Q(\omega)} K[f]$$

and

$$k_0 = (K_0[\omega] - 1)/(K_0[\omega] + 1).$$

Denote by $f_n(z)$ the extremal with respect to

* In the case when the parameter z is local, the quadratic differential φdz^2 is an invariant.

the Teichmüller mapping in the class of all quasiconformal homeomorphisms $f : U \to U'$ satisfying the conditions $f(e^{i\pi l/2^{n-1}}) = \omega(e^{i\pi l/2^{n-1}})$, $n = 1, 2, \dots;$ $l = 0, 1, \dots, 2^n - 1$. We shall denote the characteristic of the mapping $f_n(z)$ by $\mu_n(z)$ and put $|\mu_n(z)| = k_n$. Then $k_{n+1} \geq k_n$. Mappings that are limits of subsequences of $\{f_n\}$ (with respect to uniform convergence in the disk $|z| \leq 1$) will be called **limit mappings for the sequence** $\{f_n\}$.

Every limit mapping $w = f_0(z)$ has maximal dilatation $K[f_0] = K_0[\omega]$, and, consequently,

$$\lim_{n \to \infty} k_n = k_0. \tag{2}$$

Lemma. *Whatever the sequence of positive numbers $\{k_n\}$ satisfying the conditions $k_{n+1} \geq k_n$ and $\lim_{n\to\infty} k_n = k_0 < 1$, there exists a homeomorphism $\omega \in \Omega$ for which $|\mu_n(z)| = k_n$, $n = 1, 2, \dots$, and $K_0[\omega] = (1 + k_0)/(1 - k_0)$.*

Proof. Assuming, without loss of generality, that $k_1 = 0$ and $f_1(z) = z$, choose a point τ, $|\tau| = 1$, so that the conformal quadrilateral with vertices at the points $1, i, -1, \tau$, formed by the circle $|w| = 1$, has modulus equal to $(1 + k_2)/(1 - k_2)$. As $f_2(z)$ we take the mapping, extremal in the sense of Teichmüller, in the class of homeomorphisms $f : U \to U'$ satisfying the conditions $f(e^{i\pi l/2}) = e^{i\pi l/2}$, $l = 0, 1, 2;$ $f(-i) = \tau$. Assuming that the mapping $f_n(z)$ has already been constructed, we construct $f_{n+1}(z)$ as follows. If $f^*_{n+1}(z)$ is the extremal mapping in the class of homeomorphisms $f : U \to U'$ for which $f(e^{i\pi l/2^n}) = f_n(e^{i\pi l/2^n})$, $l = 0, 1, \dots, 2^n - 2$, $f(e^{i\pi(2^n-1)/2^n}) = \tau_n$, where τ_n varies continuously on the circle $|w| = 1$ from the point $w = f_n(e^{i\pi(2^n-1)/2^n})$ to $w = 1$, then $K[f^*_{n+1}]$ increases continuously from $K[f_n]$ to ∞. Therefore there exists such a $\tau_n = \tau'_n$ that $K[f^*_{n+1}] = (1+k_{n+1})/(1-k_{n+1})$, and we put $f_{n+1}(z) = f^*_{n+1}(z)$ for $\tau_n = \tau'_n$. In this case the induced homeomorphisms $\omega_n(t) = f_n|_s$ converge uniformly to some homeomorphism $\omega \in \Omega$, and $K_0[\omega] = (1 + k_0)/(1 - k_0)$. The lemma is proved.

3. Theorem. *Let $\omega \in \Omega$ and $k_0 - k_n = O(2^{-n(1+\alpha)})$, $\alpha > 0$. If there exists a limit mapping $f_0(z)$ for the sequence $\{f_n\}$ with characteristic $\mu_0(z)$, for which $|\mu_0(z)| = k_0$ almost everywhere in U, then*

$$\sup_{\varphi \in A(U),\, \|\varphi\|_{A(U)}=1} \left| \iint_U \mu_0(z)\varphi(z)\, dx\, dy \right| = k_0. \tag{3}$$

The proof of this theorem is based on the method developed in ([6]), and proceeds according to the following scheme.

Suppose there exists a limit mapping $w = f_0(z)$ for the sequence $\{f_n\}$ with characteristic $\mu_0(z)$, for which $|\mu_0(z)| = k_0$ almost everywhere in U, and let $\{f_{n_i}(z)\} \subset \{f_n(z)\}$ be a subsequence of mappings (which, for convenience, we again denote by $\{f_n\}$) converging to $f_0(z)$ uniformly in the disk $|z| \le 1$, with $f_0(z) \ne f_n(z)$, $n = 1, 2, \dots$.

In the space $L_1(U)$ of functions $\varphi(z)$ measurable in U, with norm $\|\varphi\|_{L_1(U)} = \iint_U |\varphi(z)|\, dx\, dy$, consider the functional $\mu_0(\varphi) = \iint_U \mu_0(z)\varphi(z)\, dx\, dy$. Let

$$\sup_{\varphi \in A(U),\, \|\varphi\|_{A(U)}=1} \left| \iint_U \mu_0(z)\varphi(z)\, dx\, dy \right| = k'. \tag{3'}$$

We shall show that $k' = k_0$. Suppose that $k' < k_0$. Then, by the Hahn-Banach theorem, in $L_1(U)$ there is a linear functional $m_0(\varphi)$ such that $m_0(\varphi) = \mu_0(\varphi)$ for $\varphi \in A(U)$ and $\|m_0\|_{L_1(U)} = k'$. In this case

$$m_0(\varphi) = \iint_U m_0(z)\varphi(z)\, dx\, dy, \tag{4}$$

where $m_0(z)$ is a measurable function in the disk U and $\operatorname{vrai\,max}_{z\in U} |m_0(z)| = k'$. Then for the difference $v(z) = \mu_0(z) - m_0(z)$ we shall have

$$\iint_U v(z)\varphi(z)\, dx\, dy = 0, \qquad \varphi \in A(U). \tag{5}$$

The function $v(z)$ determines the corresponding variation of the disk, i.e. the mapping $\zeta = H(z, \varepsilon)$ of the disk U onto itself with characteristic $\varepsilon v(z)$ and normalization $H(1,\varepsilon)=1$, $H(i,\varepsilon)=i$, $H(-1,\varepsilon)=-1$, representable by the formula

$$\zeta = H(z,\varepsilon) = z - \frac{\varepsilon}{\pi}\iint_U \left[\frac{v(\xi)}{\xi - z} + \frac{z^3\overline{v(\xi)}}{1 - z\bar{\xi}}\right] d\sigma(\xi) + M(z) + O(\varepsilon^2), \quad |z| \le 1, \tag{6}$$

where ε is a small real parameter, $M(z)$ is a polynomial of the form $M(z) = a + 2ibz - \bar{a}z^2$, whose coefficients are uniquely determined from the normalization conditions (see (9)).

Denoting the characteristic of the mapping $w = f_0 \circ H^{-1}(\zeta)$ by $\mu^*(\zeta)$, we obtain:

$$\tilde{\mu}(z) \equiv \mu^*(\zeta(z))\,\overline{\zeta_z}/\zeta_z = \mu_0(z) - \varepsilon v(z) + \varepsilon\overline{v(z)}\mu_0^2(z) + O(\varepsilon^2). \tag{7}$$

By applying equalities (4) and (7) it is established that, for sufficiently small $\varepsilon > 0$, the inequality

$$\sup_{\|\varphi\|_{L_1(U)}=1} \left|\iint_U \tilde{\mu}(z)\varphi(z)\,dx\,dy\right| < k_0 - O(\varepsilon) \tag{8}$$

holds, where the quantity $O(\varepsilon)$ depends only on ε, k_0, and k', i.e.

$$k^* = \operatorname{vrai}\max_{\zeta\in U} |\mu^*(\zeta)| < k_0 - O(\varepsilon). \tag{9}$$

On the other hand, by virtue of (6),

$$\zeta_{l,n} \equiv H\left(e^{i\pi l/2^{n-1}}, \varepsilon\right) = e^{i\pi l/2^{n-1}} + O(\varepsilon^2), \qquad n = 1,2,\dots;\ l = 0,1,\dots,2^n - 1,$$

and, consequently, denoting by $\tilde{\mu}_n(\zeta)$ the characteristic of the Teichmüller extremal mapping $\tilde{f}_n(\zeta)$ in the class of homeomorphisms $f : U \to U'$ satisfying the conditions $f(\zeta_{l,n}) = \omega \circ H^{-1}(\zeta_{l,n})$, $l = 0,1,\dots,2^n-1$, for fixed n, we shall have

$$|\tilde{k}_n - k_n| < O(2^n\varepsilon^2), \qquad \text{where } \tilde{k}_n = |\tilde{\mu}_n(\zeta)|. \tag{10}$$

In particular, if we take a sufficiently large $n = n_0$ and $\varepsilon = 2^{-n(1+\alpha/4)}$, then from (9) and (10) we obtain the inequality $k^* < |\mu_{n_0}(\zeta)|$, which is impossible. The contradiction obtained proves that $k' = k_0$, i.e. (3) is satisfied.

If the conditions are satisfied that ensure the existence of an element $\varphi_0 \in A(U)$, $\|\varphi_0\|_{A(U)} = 1$, for which $\sup|\mu_0(\varphi)|$ is attained on the sphere $\|\varphi\|_{A(U)} = 1$ (which certainly occurs in the case when only a finite number of points are fixed on the circles S and S'), then from (3) it follows that $\mu_0(z) = k_0\varphi_0(z)/|\varphi_0(z)|$, i.e. the mapping $f_0(z)$ is Teichmüller and, consequently, the unique extremal one in the class $Q(\omega)$.

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Note: Figure translations are in progress. See original paper for figures.