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Abstract

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MATHEMATICS

V. G. SPRINDZHUK

FINITENESS OF THE NUMBER OF RATIONAL AND ALGEBRAIC POINTS ON SOME TRANSCENDENTAL CURVES

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Studying the Diophantine properties of the values of entire functions, C. Siegel⁽¹⁾ introduced a class of functions $\varphi(z)$, which he called *E*-functions:

$$\varphi(z) = \sum_{\nu=0}^{\infty} c_{\nu} \frac{z^{\nu}}{\nu!}; \quad (1)$$

- (a) the numbers c_{ν} are algebraic and belong to a field K of finite degree over the field of rational numbers Q ; (b) for every $\varepsilon > 0$, $\overline{|c_{\nu}|} < c(\varepsilon)(\nu!)^{\varepsilon}$, where $\overline{|x|}$ for $x \in K$ denotes the maximum of the moduli of all quantities conjugate to x in K ; (c) if q_m is the least natural number for which $c_{\nu} q_m$ ($\nu = 0, 1, \dots, m$) are integers, then $q_m < c(\varepsilon)(m!)^{\varepsilon}$ for every $\varepsilon > 0$.

Siegel established⁽²⁾ the transcendence and algebraic independence of values at algebraic points of *E*-functions satisfying a system of linear homogeneous differential equations with polynomial coefficients, under the additional, difficult-to-verify condition of “normality” of the system of these equations. Siegel’s method was substantially developed by A. B. Shidlovskii^(4–6), who studied the values of *E*-functions satisfying a system of nonhomogeneous linear differential equations with polynomial coefficients, replacing Siegel’s “normality” condition by the obvious necessary condition of algebraic independence of the system of *E*-functions over the field of rational functions, thereby obtaining a final result. The Siegel-Shidlovskii theory describes the arithmetic nature of values at algebraic points of a great many entire functions ($\sin z$, $\cos z$, e^z , the Bessel function $J_0(z)$, the confluent hypergeometric function, the incomplete gamma function, Fresnel integrals, etc.). Meanwhile, it is known that any weakening of the conditions defining *E*-functions completely deprives one of the possibility of carrying out arguments by Siegel’s method. For example, the function

$$\varphi_{\lambda}(z) = \sum_{n=0}^{\infty} \frac{z^n}{(\lambda+1) \dots (\lambda+n)}, \quad \lambda \neq -1, -2, \dots, \quad (2)$$

which satisfies the differential equation $zy' + (\lambda - z)y - \lambda = 0$ and, for rational λ , belongs to the class of E -functions, for quadratic irrational λ does not belong to this class, and the arithmetic nature of its values for such λ is not decided by Siegel's method. The fact that $\varphi_\lambda(z)$, for rational $\lambda \neq -1, -2, \dots$, belongs to the class of E -functions follows from a general theorem of Siegel ⁽²⁾: the functions

$$\varphi(z) = \sum_{n=0}^{\infty} \frac{[\alpha_1, n] \dots [\alpha_l, n]}{[\beta_1, n] \dots [\beta_m, n]} z^{tn}, \quad t = m - l \geq 1, \quad (3)$$

$$[\alpha, 0] = 1, \quad [\alpha, n] = \alpha(\alpha + 1) \dots (\alpha + n - 1) \quad (n = 1, 2, \dots)$$

with rational $\alpha_j, \beta_j \neq 0, -1, -2, \dots$ belong to the class of E -functions and satisfy linear differential equations with polynomial coefficients. Siegel called these functions "hypergeometric."

Shidlovskii ^(6,7) has repeatedly drawn attention to the importance of investigating the Diophantine properties of entire functions that do not belong to the class of Siegel E -functions. In particular, quite recently Shidlovskii himself ⁽⁸⁾ established the irrationality of all, with the exception of a finite number, of the values of the functions $\varphi_\lambda(z)$, $K_\lambda(z)$, $K'_\lambda(z)$, where

$$K_\lambda(z) = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!(\lambda + 1) \dots (\lambda + n)} \left(\frac{z}{2}\right)^{2n}, \quad \lambda \neq -1, -2, \dots,$$

for algebraic λ , $z \in \mathbf{K}$, \mathbf{K} being a fixed field of algebraic numbers.

We introduce the class of E^* -functions as the class of functions of the form (1), weakening Siegel's conditions (b) and (c) (but assuming that (a) is satisfied): (b*) there exists a number a , $0 \leq a < 1$, for which $|c_\nu| < c(\varepsilon)(\nu!)^{a+\varepsilon}$, where $\varepsilon > 0$ is arbitrary; (c*) for fixed n , let q_{nm} be the smallest natural number q for which all the numbers $qc_{\nu_1}c_{\nu_2} \dots c_{\nu_n}$, $\nu_1 + \nu_2 + \dots + \nu_n \leq m$, are integers; there exists a sequence of numbers β_n , $0 \leq \beta_n = o(\sqrt{n})$ as $n \rightarrow \infty$, for which $q_{nm} < c(n, \varepsilon)(m!)^{\beta_n + \varepsilon}$ for any $\varepsilon > 0$.

It is clear that when $a = \beta_n = 0$ ($n = 1, 2, \dots$) the function $\varphi(z)$ will be an E -function, so that the class of E^* -functions contains Siegel's class of E -functions. It turns out that, applying the original Thue-Siegel idea for the construction of approximating forms, and then making a direct passage to numerical values without the construction and investigation of a system of linearly independent forms required in Siegel's method, one can obtain some results on the Diophantine properties of values of E^* -functions satisfying linear differential equations with polynomial coefficients. By this method, at the present stage of its development, it is not possible to establish the transcendence of values of E^* -functions, although assertions of the following type are proved:

(A) For any natural number d and any field of algebraic numbers \mathbf{K}_1 of finite degree over Q , there exists only a finite number of $z \in \mathbf{K}_1$ for which the function $\varphi(z)$ assumes algebraic values of degree at most d over Q . In particular, on the curve $y = \varphi(z)$ there lie no more than finitely many points $(y, z) \in \mathbf{K}_1 \times \mathbf{K}_1$.

Theorem 1. If a transcendental E^* -function $\varphi(z)$ satisfies the differential equation

$$p(z)y' + q(z)y + r(z) = 0, \quad (4)$$

where $p(z)$, $q(z)$, $r(z)$ are polynomials with algebraic coefficients, then assertion (A) holds for $\varphi(z)$.

Theorem 2. Every transcendental function of the form

$$\varphi(z) = e^{Q(z)} \left(\int_0^z R(x)e^{-Q(x)} dx + c \right),$$

where $Q(z)$, $R(z)$ are polynomials with algebraic coefficients and c is an arbitrary complex number, satisfies condition (A).

Theorem 3. Siegel hypergeometric functions (3) with algebraic parameters $\alpha_i, \beta_j \neq 0, -1, -2, \dots$ belong to the class of E^* -functions. In particular, this is true for the functions (2) with algebraic λ , so that (A) holds for them.

Theorem 1 is proved according to the following scheme. Applying Lemma II from (3), p. 67, one can, for any $h > h(n, \varepsilon)$, construct a system of polynomials $P_j(z)$ with integer coefficients from \mathbf{K} , of degree at most h , satisfying the conditions:

$$0 \neq H = \max_{0 \leq j \leq n} \overline{|P_j(z)|} < H_0^{2(\alpha + \beta n) + \varepsilon}, \quad H_0 = h^h, \quad (5)$$

function

$$f(z) = \sum_{j=0}^n P_j(z)\varphi^j(z) \quad (6)$$

at $z = 0$ has a zero of order $m \geq h[\sqrt{n}]$. Suppose that the function $f(z)$ has, at the points $z = x_i \neq 0$, zeros of orders m_i , respectively ($i = 1, 2, \dots, k$), $m_1 = \max_{(i)} m_i$. Applying the integral representation

$$f^{(m_1)}(x_1) = \frac{m_1!}{(2\pi i)^2} \int_{|z-x_1|=1} \frac{dz}{(z-x_1)^{m_1+1}} \times$$

$$\times \int_{|\xi|=R} \left(\frac{z}{\xi}\right)^m \left(\frac{z-x_1}{\xi-x_1}\right)^{m_1} \dots \left(\frac{z-x_k}{\xi-x_k}\right)^{m_k} \frac{f(\xi)}{\xi-z} d\xi$$

and choosing a suitable $R > \max_{(i)} |x_i| = \rho$, we find

$$0 \neq |f^{(m_1)}(x_1)| < H_0^{1-(\lfloor \sqrt{n} \rfloor - 1)/v + \varepsilon} m_1^{-(k/v-1)m_1 + \varepsilon}, \quad (7)$$

where $v = 1/(1 - \alpha)$ is the order of the function $\varphi(z)$, $h > h(n, \varepsilon, \rho)$. The polynomials p, q, r in equation (4) may be regarded as polynomials with integral coefficients over K , of degrees, say, $\leq s$, $|p| + |q| + |r| \leq a$. Define the polynomials $P_{jl} = P_{jl}(z)$ by the equalities $P_{j0} = P_j$ ($j = 0, 1, \dots, n$), where the P_j are determined by conditions (5), (6), and P_{jl} ($l = 1, 2, \dots$) by the equalities

$$P_{0l+1} = pP'_{0l} - rP_{1l},$$

$$P_{jl+1} = pP'_{jl} - jqP_{jl} - (j+1)rP_{j+1,l} \quad (j = 1, 2, \dots, n-1),$$

$$P_{nl+1} = pP'_{nl} - nqP_{nl}.$$

Then

$$\max_{0 \leq j \leq n} |P_{jl}| < a^{lH} (h + sl + 2n + 1)^l. \quad (8)$$

Putting $f_0(z) = f(z)$, $f_{l+1}(z) = p(z)f'_l(z)$, we find

$$f_l(z) = \sum_{j=0}^n P_{jl}(z)\varphi^j(z) \quad (l = 0, 1, \dots), \quad f_{m_1}(x_1) = p^{m_1}(x_1)f^{(m_1)}(x_1).$$

If we assume that the numbers x_1, x_2, \dots, x_k are algebraic and belong to the field K_1 , then, in view of (5), (7), (8), we can establish (A), if n is taken sufficiently large and it is assumed that k can be sufficiently large.

Passing from equation (4) to the system of equations

$$y'_l = \sum_{j=1}^m Q_{jl}y_j + Q_l \quad (l = 1, 2, \dots, m), \quad (9)$$

where Q_{jl}, Q_l are rational functions with algebraic coefficients, one can prove, for a system of E^* -functions, assertions of the following type:

- (B) For any natural number d and any field of algebraic numbers K_1 of finite degree over Q , there exists only a finite number of $z \in K_1$ for which the functions $\varphi_1(z), \dots, \varphi_m(z)$ simultaneously take algebraic values of degrees not exceeding d over Q . In particular, on the curve $\Gamma = (\varphi_1(z), \dots, \varphi_m(z))$ there lie no more than a finite number of points from K_1^m for $z \in K_1$.

Theorem 4. If the E^* -functions $\varphi_1(z), \dots, \varphi_m(z)$ satisfy the system of equations (9) and are algebraically independent over the field of rational

functions $C(z)$, C being the field of complex numbers, then they satisfy condition (B).

Theorem 5. If all the coefficients of equation (9) are polynomials, then for any system of functions $\varphi_1(z), \dots, \varphi_m(z)$ satisfying (9) and algebraically independent over $C(z)$, assertion (B) is valid.

The proof of Theorem 4 is carried out according to the same scheme as the proof of Theorem 1, and contains only technical complications. The proof of Theorem 2 and of the more general Theorem 5 is based on the following initial remarks. If the functions $\varphi_1(z), \dots, \varphi_m(z)$ at an algebraic point \varkappa take algebraic values simultaneously, then, by virtue of (9), we can establish that the functions $\varphi_1(z + \varkappa), \dots, \varphi_m(z + \varkappa)$ belong to the class E^* . For the proof of (6*) it should be noted that equation (9) is satisfied only by entire functions of finite order (by the condition of Theorem 5, Q_{jl}, Q_l are polynomials), and the proof of (B*) is obtained by successive differentiation of (9), with simple estimates. Then we apply Theorem 4.

Finally, both the membership of Siegel hypergeometric functions with algebraic parameters in the class of E^* -functions, and the nonmembership of the function (2) with quadratic irrationality in the class of E^* -functions, are proved by the application of p -adic analysis.

Institute of Mathematics
Academy of Sciences of the BSSR

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